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UNIFORMLY CONVERGENT NONCONFORMING TETRAHEDRAL ELEMENT FOR DARCY-STOKES PROBLEM*

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Abstract

In this paper, we construct a tetrahedral element named DST20 for the three dimensional Darcy-Stokes problem, which reduces the degrees of velocity in [30]. The finite element space $\boldsymbol{V}_h$ for velocity is $\boldsymbol{H}(\mathrm{div})$-conforming, i.e., the normal component of a function in $\boldsymbol{V}_h$ is continuous across the element boundaries, meanwhile the tangential component of a function in $\boldsymbol{V}_h$ is average continuous across the element boundaries, hence $\boldsymbol{V}_h$ is $\boldsymbol{H}^1$-average conforming. We prove that this element is uniformly convergent with respect to the perturbation constant ε for the Darcy-Stokes problem. At the same time, we give a discrete de Rham complex corresponding to DST20 element.

Mathematics subject classification: 65N15, 65N30.
Key words: Darcy-Stokes problem, Mixed finite elements, Tetrahedral element, Uniformly convergent.

1. Introduction

In this paper, we consider the mixed finite element methods for the following singular perturbation problem of three dimension [12,30]:

$$\begin{cases} (I-\varepsilon^2\boldsymbol{\Delta})\boldsymbol{u}-\mathbf{grad}p=\boldsymbol{f} & \text{in } \boldsymbol{\Omega}, \\ \mathrm{div}\boldsymbol{u}=0 & \text{in } \Omega, \\ \boldsymbol{u}=0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega\subset\mathbb{R}^3$ is a bounded, convex and connected polygonal domain with boundary $\partial\Omega$, $\varepsilon\in(0,1]$ is a parameter, $\boldsymbol{\Delta}$ is the standard Laplace operator. The vector field $\boldsymbol{u}$ and the scalar field p are corresponding to velocity and pressure in flow problems, respectively.

The problem (1.1) admits a unique solution and p is determined only up to addition of a constant [22]. When ε is not too small, this problem is simply a standard Stokes problem, but with an additional non-harmful lower order term. If $\boldsymbol{f}=0$ and ε approaches zero, the problem (1.1) tends to a mixed formulation of the Poisson equation with homogeneous Neumann boundary conditions i.e. a Darcy flow. When $\varepsilon=0$, the first equation of (1.1) has the form of Darcy's law for flow in a homogeneous porous medium. Generalizations of the system (1.1) have been proposed in various physical models, see, e.g., [14,15,17,21,23,24,31,33].

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In order to make the discrete problem using mixed finite element method for (1.1) well posed, one has to be careful to choose the velocity/pressure finite element spaces. One of the usual methods([6] etc.) chooses nonconforming Crouzeix-Raviart elements [13] that are convergent for Stokes problem, and we also know that Raviart-Thomas elements that are convergent for the mixed two order problem [28], are not uniformly convergent with respect to the perturbation constant ε. Several methods are presented to construct uniformly convergent elements for (1.1). The first method uses H^1-conforming elements for velocity but on the special meshes, such as [3,27,29,32,34]. The second method is stabilized method based on different approaches, such as [4,8,9,18–20,31]. The third method uses $H(\mathrm{div},\Omega)$-conforming but H^1-nonconforming elements [10,16,22,35].

In three dimension case, the bubble function method proposed in [11] for 3D fourth-order elliptic problem can also be employed in the construction of uniformly convergent finite elements for the Darcy-Stokes problem. Tai & Wither, 2006, [30] presented a $H(\mathrm{div})$-conforming and uniformly convergent tetrahedron element with 24 degrees of freedom for velocity. In this paper, we present a $H(\mathrm{div})$-conforming and uniformly convergent tetrahedron element with 20 degrees of freedom for velocity. We name the element DSC20 element. Another object of this paper is to construct the discrete de Rham complex corresponding to DSC20 element. Discrete de Rham complex are fundamental tools in the construction of stable elements for some finite element methods [1,2]. Well-known examples of such finite element spaces are described in [25,26]. In three space dimensions the Sobolev space version of the de Rham complex can be written in the form

$$R \xrightarrow{\subset} H^2 \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0.$$

A corresponding discrete de Rham complex is the form

$$R \xrightarrow{\subset} S_h \xrightarrow{\mathbf{grad}} \boldsymbol{W}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0,$$

where S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$ and Q_h are conforming or nonconforming finite element spaces of $\boldsymbol{H}^2(\Omega)$, $\boldsymbol{H}(\mathbf{curl},\Omega)$, $\boldsymbol{H}(\mathrm{div},\Omega)$ and $L^2(\Omega)$, respectively.

In our discrete de Rham complex, S_h is H^1-conforming and H^2-average conforming, it is convergent for the fourth order elliptic problem and uniformly convergent for the fourth order singular perturbation problem; $\boldsymbol{W}_h$ is $\boldsymbol{H}(\mathbf{curl})$-conforming and H^1-average conforming, $\boldsymbol{V}_h$ presented in this paper is $\boldsymbol{H}(\mathrm{div})$-conforming and H^1-average conforming, it is uniformly convergent for Darcy-Stokes singular perturbation problem.

The rest of this paper is organized as follows. In section 2, we introduce the notation and some well-known results of the Darcy-Stokes problem presented in [22]. The construction of DSC20 element is given in section 3. In section 4, we discuss the uniform convergence and the uniform error estimates of the discrete Darcy-Stokes problem. The last section, we construct a discrete de Rham complex corresponding to DST20 elements.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a convex and bounded polygon, $H^m(\Omega)$ and $H_0^m(\Omega)$ be the usual Sobolev spaces with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$ respectively, $H^{-m}(\Omega)$ be the dual space of $H_0^m(\Omega)$, $L_0^2(\Omega)$ be the space of $L^2(\Omega)$ functions with mean value zero. Bold-faces are used to denote the vector functions.

The differential operators are defined as the following: