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APPROXIMATIONS OF HYPERSINGULAR INTEGRALS FOR NEGATIVE FRACTIONAL EXPONENT*

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Abstract

This article presents approximations of the hypersingular integrals $\int_a^b g(x)(x-t)^\alpha dx$ and $\int_a^b g(x)|x-t|^\alpha dx$ with arbitrary singular point $t \in (a,b)$ and negative fraction number $\alpha < -1$. These general expansions are applicable to a large range of hypersingular integrals, including both popular hypersingular integrals discussed in the literature and other important ones which have not been addressed yet. The corresponding mid-rectangular formulas and extrapolations, which can be calculated in fairly straightforward ways, are investigated. Numerical examples are provided to illustrate the features of the numerical methods and verify the theoretical conclusions.

Mathematics subject classification: 65B15, 42B20.
Key words: Hypersingular integral, Negative fractional exponent, Mid-rectangular quadrature formula, Extrapolation.

1. Introduction

In recent years, it has attracted attention to approximate hypersingular integrals [34, 41] arising from different research areas, such as oil engineering,acoustics, electromagnetics, fracture mechanics, fluid dynamics, heat conduction and elasticity [8, 11, 15, 17, 18, 20–22, 24, 26, 31, 43, 45, 46, 49, 50, 53, 54]. Furthermore, approximations for hypersingular integrals are often needed in order to construct the numerical algorithms for solving hypersingular integral equations [30], which have been utilized to study many real world problems, see [3–7, 28, 29, 32, 33, 37–40, 52] and the references cited therein.

In order to evaluate the hypersingular integrals, different numeric techniques have been developed, such as interpolation polynomials, splines, segments of orthogonal series, etc. [9, 23, 25, 27, 44, 47, 48]. Furthermore, Sidi [42] discussed the Euler-Maclaurin expansions for integrals with arbitrary algebraic endpoint singularities, and Boykov et al. [1, 2] presented optimal, asymtotically optimal and optimal in order algorithms for numerical evaluation of hypersingular integrals with fixed and varying singularities. The direct method [11, 14], the method of refinable operators [12] and other methods [46, 51] for the evaluation of singular integrals have also been developed.

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The Euler-Maclaurin expansions,mid-rectangular quadrature formulas and extrapolations of the following hypersingular integrals are discussed deliberately in [16].

$$I(f) = \int_a^b g(x)(x-t)^m dx, \tag{1.1}$$

$$I(f) = \int_a^b g(x)|x-t|^m dx, \tag{1.2}$$

where m is a negative integer and $t \in (a,b)$.

Nevertheless, we have seldom found literatures which discuss the integrals with fractional singularity as follows.

$$I_1(f) = \int_a^b g(x)(x-t)^\alpha dx, \tag{1.3}$$

$$I_2(f) = \int_a^b g(x)|x-t|^\alpha dx, \tag{1.4}$$

where α is a negative fraction and $t \in (a,b)$.

In most cases, Euler-Maclaurin expansions of Hypersingular integrals with negative fractional exponent concludes negative exponent of h, so that many quadrature formulas including the mid-rectangular formula are divergent. We have to use Richardson extrapolations to get convergent results.

In this article, we will derive Euler-Maclaurin expansions of the general hypersingular integrals $\int_a^b g(x)|x-t|^\alpha dx$ and $\int_a^b g(x)(x-t)^\alpha dx$ for negative fraction α and arbitrary singularity point $t \in (a,b)$. Then the corresponding mid-rectangular formulas and extrapolations will be constructed. The rest of this paper is organized as follows: in section 2, a general definition of Hadamard finite-part integrals is recalled; in section 3, we present the asymptotic expansions, quadrature formulas and their extrapolations for (1.3) and (1.4); in section 4, numerical examples are provided to illustrate the features of the numerical methods and verify the theoretical conclusions.

2. Definition of Hadamard Finite-Part Integrals

For a function $f(x)$, which could be hypersingular nearby the origin of coordinates, the Hadamard finite-part integral is defined as follows.

Definition 2.1. ([36]) *Let $f(x)$ be integrable over (ϵ, b) for any ϵ satisfying $0 < \epsilon < b < \infty$. Suppose that there exists a strictly monotonic increasing sequence $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$, and a non-negative integer J such that the following expansion*

$$\int_\varepsilon^b f(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{J} I_{i,j}\varepsilon^{\alpha_i}\ln^j(\varepsilon) \tag{2.1}$$

converges for any $\varepsilon \in (0,h)$ with some $h > 0$. Then the Hadamard finite-part integral of (2.1) *is defined as follows:*

$$f.p.\int_0^b f(x)dx = \begin{cases} 0 & if\ \alpha_i \neq 0\ for\ all\ i, \\ I_{i,0} & if\ \alpha_i = 0\ for\ some\ i. \end{cases} \tag{2.2}$$