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ON DOUBLY POSITIVE SEMIDEFINITE PROGRAMMING RELAXATIONS*

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Abstract

Recently, researchers have been interested in studying the semidefinite programming (SDP) relaxation model, where the matrix is both positive semidefinite and entry-wise nonnegative, for quadratically constrained quadratic programming (QCQP). Comparing to the basic SDP relaxation, this doubly-positive SDP model possesses additional $O(n^2)$ constraints, which makes the SDP solution complexity substantially higher than that for the basic model with $O(n)$ constraints. In this paper, we prove that the doubly-positive SDP model is equivalent to the basic one with a set of valid quadratic cuts. When QCQP is symmetric and homogeneous (which represents many classical combinatorial and non-convex optimization problems), the doubly-positive SDP model is equivalent to the basic SDP even without any valid cut. On the other hand, the doubly-positive SDP model could help to tighten the bound up to 36%, but no more. Finally, we manage to extend some of the previous results to quartic models.

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Key words: Doubly nonnegative matrix, Semidefinite programming, Relaxation, quartic optimization.

1. Introduction

Consider the quadratically constrained quadratic programming problem

$$\begin{aligned} \text{Maximize} \quad & x^T Q_0 x + c_0^T x \\ \text{Subject to} \quad & x^T Q_i x + c_i^T x = b_i, \quad i = 1, \ldots, m, \\ & -e \le x \le e, \end{aligned} \tag{1.1}$$

where symmetric matrix $Q_i \in \Re^{n\times n}$ and vector $c_i \in \Re^n$, $i = 0, 1, \cdots, m$, and $e \in \Re^n$ is the vector of all ones. Note that any other lower and upper bounds on decision variables, $l \le x \le u$, can be transformed to $-e \le x \le e$ through scaling and linear translation. Also, the results developed in this paper are easily extendable to quadratic inequality constraints. We assume

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that the QP problem is known to be feasible, so that any of its relaxation models would be also feasible.

The classical and basic semidefinite programming relaxation for problem (1.1) is

$$\begin{aligned} \text{Maximize} \quad & Q_0 \cdot X + c_0^T x \\ \text{Subject to} \quad & Q_i \cdot X + c_i^T x = b_i, \quad i = 1, \cdots, m, \\ & X_{jj} \le 1, \quad j = 1, \cdots, n, \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0. \end{aligned} \tag{1.2}$$

If the SDP solution has a rank one property, that is, $X^* = x^*(x^*)^T$, then x^* solves problem (1.1).

Recently, there are research efforts to construct stronger or tighter SDP relaxations for QCQP. One particular effort is to let $y = (x + e)/2$ so that problem (1.1) has an equivalent form within the nonnegative domain:

$$\begin{aligned} \text{Maximize} \quad & 4y^T Q_0 y + (2c_0^T - 4e^T Q_0)y + e^T Q_0 e - c_0^T e \\ \text{Subject to} \quad & 4y^T Q_i y + (2c_i^T - 4e^T Q_i)y + e^T Q_i e - c_i^T e = b_i, \quad i = 1, \ldots, m, \\ & 0 \le y \le e. \end{aligned} \tag{1.3}$$

Using the knowledge that all decision variables need to be nonnegative, the following SDP relaxation can be constructed:

$$\begin{aligned} v_p^* := \quad \text{Maximize} \quad & 4Q_0 \cdot Y + (2c_0^T - 4e^T Q_0)y + e^T Q_0 e - c_0^T e \\ \text{Subject to} \quad & 4Q_i \cdot Y + (2c_i^T - 4e^T Q_i)y + e^T Q_i e - c_i^T e = b_i, \quad i = 1, \ldots, m, \\ & Y_{jj} \le y_j \le 1, \ j = 1, 2, \cdots, n, \\ & Z := \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \succeq 0, \\ & Y_{ij} \ge 0, \ \forall 1 \le i < j \le n. \end{aligned} \tag{1.4}$$

Since $Z \succeq 0$ as well as $Z \ge 0$, it is called a *doubly-positive semidefinite program*; e.g., see Dong et al. [5], Burer [3] and Burer et al. [4]. It is well known that there exists a hierarchy of linear and semidefinite representable cones that approximate the co-positive and completely positive cone (see Bomze et al. [2] and Parrilo [13]), where the doubly-positive SDP is a mostly used relaxation technique due to its computability. Very recent research has discussed its applications in many areas, e.g., appointment scheduling by Kong et al. [10], order statistics by Natarajan et al. [11].

The doubly-positive SDP increases the number of constraints from $m + n$ in basic SDP model (1.2) to $m + 2n + n(n-1)/2$ in (1.4). With such a sacrifice in computational complexity, (1.4) must be stronger or tighter than (1.2). In this paper, we are trying to answer this very question: when and how much is the doubly-positive SDP relaxation tighter than the basic SDP one?

Besides, in the last section, we manage to extend the doubly-positive relaxation to the quartic optimization, which has wide applications in sensor network localization [1], portfolio management with high moments information [9] and et. al., and obtain some results which are similar to the quadratic optimization.