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A FIRST-ORDER NUMERICAL SCHEME FOR FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS IN BOUNDED DOMAINS*

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Abstract

We propose a novel numerical scheme for decoupled forward-backward stochastic differential equations (FBSDEs) in bounded domains, which corresponds to a class of nonlinear parabolic partial differential equations with Dirichlet boundary conditions. The key idea is to exploit the regularity of the solution (Y_t, Z_t) with respect to X_t to avoid direct approximation of the involved random exit time. Especially, in the one-dimensional case, we prove that the probability of X_t exiting the domain within Δt is on the order of $\mathcal{O}((\Delta t)^{\varepsilon}\exp(-1/(\Delta t)^{2\varepsilon}))$, if the distance between the start point X_0 and the boundary is at least on the order of $\mathcal{O}((\Delta t)^{\frac{1}{2}-\varepsilon})$ for any fixed $\varepsilon > 0$. Hence, in spatial discretization, we set the mesh size $\Delta x \sim \mathcal{O}((\Delta t)^{\frac{1}{2}-\varepsilon})$, so that all the interior grid points are sufficiently far from the boundary, which makes the error caused by the exit time decay *sub-exponentially* with respect to Δt. The accuracy of the approximate solution near the boundary can be guaranteed by means of high-order piecewise polynomial interpolation. Our method is developed using the implicit Euler scheme and cubic polynomial interpolation, which leads to an overall first-order convergence rate with respect to Δt.

Mathematics subject classification: 60H35, 60H10, 65C20, 65C30.
Key words: Forward-backward stochastic differential equations, Exit time, Dirichlet boundary conditions, Implicit Euler scheme.

1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ for $T > 0$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$, generated by the m-dimensional standard Brownian motion $W_t := (W_t^1, \ldots, W_t^m)^\top$. We are interested in numerical solution of the following decoupled forward-backward stochastic

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differential equation (FBSDE), defined in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, i.e.,

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, & \text{(SDE)}, \\ Y_t = \varphi(T\wedge\tau, X_{T\wedge\tau}) + \int_{t\wedge\tau}^{T\wedge\tau} f(s, X_s, Y_s, Z_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s, & \text{(BSDE)}, \end{cases} \tag{1.1}$$

where $\tau := \inf\{t > 0, X_t \notin D\}$ is the first exit time of (t, X_t) from a cylindrical domain $[0,T]\times D \subset [0,+\infty)\times\mathbb{R}^d$ for an open piecewise smooth connected set D, and the initial condition X_0 is in the domain D. We assume that $\mathbb{P}(\tau < \infty) = 1$. The functions $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T]\times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^q\times\mathbb{R}^{q\times m} \to \mathbb{R}^q$ are referred to as the drift coefficient, the diffusion coefficient and the generator, respectively. The two stochastic integrals with respect to W_t are of the Itô type. $(X_t, Y_t, Z_t) : \Omega\times[0,T] \to \mathbb{R}^d\times\mathbb{R}^q\times\mathbb{R}^{q\times m}$ are the unknowns of the FBSDE in (1.1). A triple (X_t, Y_t, Z_t) is called an L^2-adapted solution of the FBSDE, if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies (1.1).

Pardoux and Peng [1] first proved the existence and uniqueness of nonlinear backward stochastic differential equation (BSDEs) with deterministic terminal time, under the assumption that f is uniformly Lipschitz in Y_t and Z_t. The well-posedness of FBSDE (1.1) with random terminal time has been investigated in [2, 3]. It is well known that the FBSDEs of interest are closely connected to a class of nonlinear partial differential equations (PDEs) [4–7]. This relationship, also known as the nonlinear Feynman-Kac theory, is the theoretical foundation of this work. Nonlinear second-order PDEs arise from many fields in science and engineering such as astrophysics, differential geometry, image processing, mathematical finance, etc. Besides the need for developing deep and sophisticated analytical methods for analyzing this class of PDEs, there is an ever increasing demand for efficient and reliable numerical methods for computing their solutions. One of the disadvantages of existing numerical methods, such as finite element and finite difference methods, is the complexity and robustness of the involved linear and nonlinear iterative solvers. Thus, our goal is to develop an accurate and efficient numerical scheme for the FBSDE in (1.1), and utilize the developed scheme to solve the nonlinear parabolic PDEs with Dirichlet boundary conditions.

In the literature, not many works have been devoted to numerical approximation of FBSDEs in bounded domains, but we would like to mention [8–11]. The main issue in solving (1.1) is the low accuracy of the approximate solution near the Dirichlet boundary due to the involvement of the exit time τ. Among those works, there are basically two types of techniques to deal with the exit time. In [10], the approximate exit time, e.g., associated with the discretized forward SDE, are directly used in numerical schemes, so that the error between the true exit time and the approximate one enters the global error. This type of methods usually require weaker assumptions on the coefficients of the FBSDEs, and achieve at most half-order convergence $\mathcal{O}((\Delta t)^{1/2})$ even in the weak sense. The second type of methods, e.g., the works [8, 9, 11], exploit the nonlinear Feynman-Kac formula to construct special numerical schemes for solving the FBSDEs near the boundary without directly approximating the exit time. Under sufficient assumptions on the coefficients in (1.1) and the geometry of the domain D, the numerical schemes proposed in [8] can achieve first-order convergence.

The numerical schemes presented in this paper conceptually belong to the second type; the main idea is to exploit the smoothness of (Y_t, Z_t) with respect to X_t to avoid direct approximation of the exit time τ. Specifically, it is known that the probability $\mathbb{P}(\tau \le \Delta t)$, for a small $\Delta t > 0$, decays very fast as the starting point X_0 moves away from ∂D towards the center of