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EIGENVALUES OF THE NEUMANN-POINCARÉ OPERATOR FOR TWO INCLUSIONS WITH CONTACT OF ORDER m: A NUMERICAL STUDY*

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Abstract

In a composite medium that contains close-to-touching inclusions, the pointwise values of the gradient of the voltage potential may blow up as the distance δ between some inclusions tends to 0 and as the conductivity contrast degenerates. In a recent paper [9], we showed that the blow-up rate of the gradient is related to how the eigenvalues of the associated Neumann-Poincaré operator converge to $\pm\frac{1}{2}$ as $\delta \to 0$, and on the regularity of the contact. Here, we consider two connected 2-D inclusions, at a distance $\delta > 0$ from each other. When $\delta = 0$, the contact between the inclusions is of order $m \geq 2$. We numerically determine the asymptotic behavior of the first eigenvalue of the Neumann-Poincaré operator, in terms of δ and m, and we check that we recover the estimates obtained in [10].

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1. Eigenvalues of the Neumann-Poincaré Operator for two Inclusions

Let $D_1, D_2 \subset \mathbb{R}^2$ be two bounded, smooth inclusions separated by a distance $\delta > 0$. We assume that D_1 and D_2 are translates of two reference touching inclusions

$$D_1 \;=\; D_1^0 + (0, \delta/2), \quad D_2 \;=\; D_2^0 + (0, -\delta/2).$$

We assume that D_1^0 lies in the lower half–plane $x_1 < 0$, D_2^0 in the upper half–plane, and that they meet at the point 0 tangentially to the x_1–axis (see Figure 1.1). We make the following additional assumptions on the geometry:

A1. The inclusions D_1^0 and D_2^0 are strictly convex and only meet at the point 0.

A2. Around the point 0, ∂D_1^0 and ∂D_2^0 are parametrized by 2 curves $(x, \psi_1(x))$ and $(x, -\psi_2(x))$ respectively. The graph of ψ_1 (resp. ψ_2) lies below (resp. above) the x–axis.

A3. The boundary ∂D_i^0 of each inclusion is globally $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha \leq 1$.

A4. The function $\psi_1(x) + \psi_2(x)$ is equivalent to $C|x|^m$ as $x \to 0$, where $m \geq 2$ is a fixed integer and C is a positive constant.

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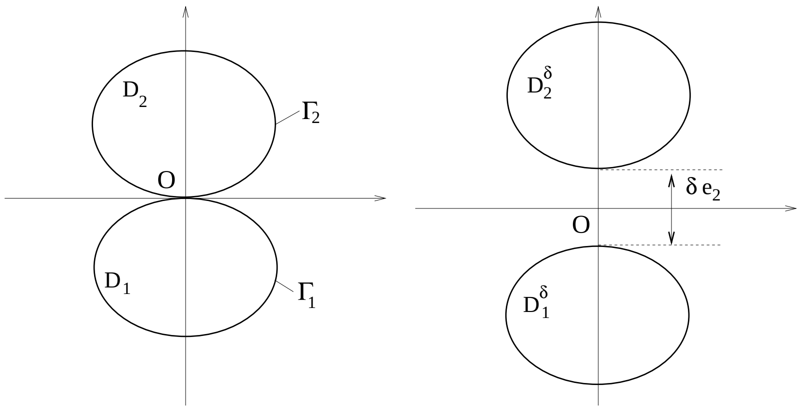

Fig. 1.1. The touching and non–touching configurations.

Let $a(X)$ be a piecewise constant function that takes the value $0 < k \neq 1$ in each inclusion and 1 in $\mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$, that is

$$a(X) = 1 + (k-1)\chi_{D_1 \cup D_2}(X),$$

where $\chi_{D_1 \cup D_2}$ is the characteristic function of $D_1 \cup D_2$. Given a harmonic function H, we denote u the solution to the PDE

$$\begin{cases} \operatorname{div}(a(X)\nabla u(X)) = 0 & \text{in } \mathbb{R}^2 \\ u(X) - H(X) \to 0 & \text{as } |X| \to \infty. \end{cases} \tag{1.1}$$

Since H is harmonic in the whole space the regularity of u at a fixed value k, only depends on the smoothness of the inclusions and of their distribution [15].

One can express u in terms of layer potentials [1, 22]

$$u(X) = S_1\varphi_1(X) + S_2\varphi_2(X) + H(X), \tag{1.2}$$

where S_i denotes the single layer potential on ∂D_i, defined for $\varphi \in H^{-1/2}(\partial D_i)$ by

$$S_i\varphi(X) = \frac{1}{2\pi}\int_{\partial D_i} \ln|X-Y|\,\varphi(Y)\,d\sigma(Y).$$

Denoting the conductivity contrast by

$$\lambda = \frac{k+1}{2(k-1)} \in \Big(-\infty, -\frac{1}{2}\Big) \cup \Big(\frac{1}{2}, +\infty\Big)$$

and expressing the transmission conditions satisfied by u, one sees that the layer potential $\varphi = (\varphi_1, \varphi_2) \in H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial D_2)$ satisfies the system of integral equations

$$(\lambda I - K_\delta^*)\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \partial_{\nu_1} H_{|\partial D_1} \\ \partial_{\nu_2} H_{|\partial D_2} \end{pmatrix}, \tag{1.3}$$

where $\nu_i(X)$ denotes the outer normal at a point $X \in \partial D_i$. The operator K_δ^* is the Neumann-Poincaré operator for the system of two inclusions

$$K_\delta^*\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} K_1^* & \partial_{\nu_1} S_{2|\partial D_1} \\ \partial_{\nu_2} S_{1|\partial D_2} & K_2^* \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1.4}$$