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INVERSION OF THE BREMMER SERIES*

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Abstract

We consider the inverse backscattering problem for scalar waves in one dimension. We analyze the convergence of the inverse Bremmer series in this context and study its use in numerical simulations.

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1. Introduction

The inverse scattering problem (ISP) consists of recovering the spatially varying scattering potential of a bounded medium from measurements recorded on its boundary. In a typical experiment, a source creates a wave that is incident on the medium and the resulting scattered wave is collected by an array of detectors. There are many variants of this basic scenario. The source may be pulsed or time harmonic and the detector may be time or frequency resolved, polarization or phase sensitive, located in the near- or far-field and so on. There are numerous areas of application of the ISP ranging from seismic imaging of the earth to optical or ultrasonic imaging of the human body. We note that there is a considerable body of work on the ISP that has been been reviewed in [1–3]. Much is known about theoretical aspects of the problem, especially on matters of uniqueness, stability and partial data. There has also been significant effort devoted to the development of computational techniques to reconstruct the scattering potential. These include optimization methods, qualitative methods such as the linear sampling method, and direct approaches such as the $\bar{\partial}$-method and inversion of the Born series. The inverse Born series has been applied to inverse problems associated with the diffusion equation [4,5], radiative transport equation [6] and the scalar wave equation [7]. In combination with a spectral method for solving the linear inverse problem, the inverse Born series leads to a fast image reconstruction algorithm with analyzable convergence, stability and error.

In this paper, we investigate the inverse of the Bremmer series for scalar waves. In contrast to the Born series, the Bremmer series allows for a directional decomposition of the wave field into up- and down-going components [8]. This decomposition is natural in the setting of

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seismic imaging, where the Earth's subsurface is layered in the vertical direction. Inversion of the Bremmer series was proposed as a direct method for solving the ISP in [9]. Here we analyze the convergence, stability and approximation error of the method. We also illustrate its use in numerical simulations. We find that the series converges rapidly for low contrast objects. As the contrast is increased, the higher order terms systematically improve the reconstructions until, at sufficiently large contrast, the series diverges. We note that our analysis is restricted to one-dimension. The higher-dimensional case has a different mathematical structure and will be discussed elsewhere.

The remainder of this paper is organized as follows. In Section 2 we construct the Bremmer series for time-harmonic scalar waves in one dimension. We then derive various estimates that are used to study the convergence of the Bremmer series and its inverse. Inversion of the Bremmer series is discussed in Section 3. Numerical reconstructions are presented in Section 4. Finally, our conclusions are presented in Section 5.

2. Forward Problem

We begin by considering the one-dimensional wave equation on the real line for time-harmonic waves. The field u obeys

$$\partial_x^2 u + k_0^2(1+\eta(x))u = 0, \tag{2.1}$$

where k_0 is the wave number and the scattering potential η is supported on the interval $[0,a]$. It will prove useful to decompose the field into the sum of an incident field and a scattered field:

$$u = u_0 + u_s. \tag{2.2}$$

The incident field will be taken to be a plane wave of the form

$$u_0(x) = e^{ik_0x}. \tag{2.3}$$

The scattered field u_s satisfies

$$\partial_x^2 u_s + k_0^2 u_s = -k_0^2\eta(x)u \tag{2.4}$$

and obeys the Sommerfeld radiation condition

$$\lim_{|x|\to\infty}(\partial_x u_s - ik_0u_s) = 0. \tag{2.5}$$

The forward problem is to determine u for a given η and incident field.

In order to decompose the field u into its up- and down-going components, we express (2.1) in matrix form as

$$\begin{cases} (I\partial_x - A)\vec{U}' = 0, \\ (I\partial_x - A_0)\vec{U}_0' = 0, \end{cases} \tag{2.6}$$

where I is the 2×2 identity matrix,

$$A = \begin{pmatrix} 0 & 1 \\ -k_0^2(1+\eta(x)) & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 1 \\ -k_0^2 & 0 \end{pmatrix}. \tag{2.7}$$

$$\vec{U}' = \begin{pmatrix} u \\ \partial_x u \end{pmatrix}, \quad \vec{U}_0' = \begin{pmatrix} u_0 \\ \partial_x u_0 \end{pmatrix}. \tag{2.8}$$