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CONSTRUCTION OF GPT-VANISHING STRUCTURES USING SHAPE DERIVATIVE*

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Abstract

The Generalized Polarization Tensors (GPT) is a series of tensors which contain informations on the shape of a domain and its material parameters. The aim of this paper is to provide a method of constructing GPT-vanishing structures using shape derivative for two-dimensional conductivity or anti-plane elasticity problem. We assume a multi-coating geometry as a candidate of GPT-vanishing structure. We define a cost functional to minimize GPT and compute the shape derivative of this functional deriving an asymptotic expansion of the perturbations of the GPTs due to a small deformation of interfaces of the structure. We present some numerical examples of GPT-vanishing structures for several different shaped inclusions.

Mathematics subject classification: 65N06, 65B99.
Key words: Generalized polarization tensor, Asymptotic expansions, Shape derivative.

1. Introduction

With a domain with material parameter, an infinite number of tensors, called the Generalized Polarization Tensors (GPT), is associated. The notion appears naturally when we consider the multipolar expansion of the perturbation of the electrical potential due to the presence of an inclusion. It gives us stratified informations on the material, having properties like symmetry and positivity. It is known that the full set of GPT determines the shape uniquely [2]. Recently, there have been works on various applications of GPT in inverse problems and effective medium theory such as detection of small inhomogeneities, shape description, derivation of effective properties of composites, and so on. For these applications see [3] and references therein.

To mathematically introduce the concept of GPT, we consider the conductivity problem in $\mathbb{R}^2$:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } \mathbb{R}^2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

for a harmonic function H in $\mathbb{R}^2$ and a conductivity σ given by

$$\sigma = k\chi(\Omega) + \chi(\mathbb{R}^2 \setminus \Omega),$$

where χ is the characteristic function.

Here we assume that Ω is a bounded Lipschitz domain with $0 \in \Omega$ and $k \neq 1$. Then the perturbation $u - H$ is given by the multipolar expansion [2]:

$$(u - H)(x) = \sum_{|\alpha|,|\beta|=1}^{+\infty} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial^\alpha \Gamma(x) M_{\alpha\beta} \partial^\beta H(0) \quad \text{as } |x| \to +\infty, \tag{1.2}$$

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where Γ is the fundamental solution to the Laplacian. Here and throughout this paper $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ are multi-indices and $|\alpha| = \alpha_1 + \alpha_2$.

The tensor $M_{\alpha\beta}$ is called the generalized polarization tensor (GPT). When $|\alpha| = |\beta| = 1$, we call it the polarization tensor (PT). Formula (1.2) shows that through GPTs we have complete information about the far-field expansion of the perturbation $u - H$.

The idea of vanishing the PT of inclusions was successfully used in effective medium approximation. Bruggemann calculated the effective conductivity of an aggregate comprised of grains assuming that the PT of a representative sample of grains vanishes when they are embedded in the medium of effective conductivity [12]. Christensen and Lo obtained the effective shear modulus of an elastic composite using vanishing condition of (elastic) PT of coated sphere embedded in the effective medium [13]. For more materials, refer to [19] and references therein. Recently, Ammari et al. [7] used the idea of vanishing GPTs to design cloaking structures efficiently.

An inclusion Ω with conductivity σ is called a GPT-vanishing structure of order N if

$$\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(\sigma) = 0$$

for all a_α, b_β such that $\sum_\alpha a_\alpha x^\alpha$ and $\sum_\beta b_\beta x^\beta$ are harmonic polynomials of degree less than N. Then we see from (1.2) that the solution u to (1.1) satisfies

$$u(x) - H(x) = O(|x|^{-N-1})$$

as $|x| \to \infty$ for any harmonic polynomial H of degree N or less. In [7] it is shown that if we use the GPT-vanishing structure combined with blow-up transformation to design a new near-cloaking structure, then the effect of near-cloaking is significantly enhanced. See also [5, 8, 9] for further development to Helmholtz and Maxwell's equations.

In this paper, we consider the problem of constructing GPT-vanishing structure coating a given inclusion of general shape. If we consider the geometry of concentric circles, GPT-vanishing structures of any order can be calculated solving a system of algebraic equations as in [7]. For general shape, we introduce a functional on GPT and calculate its shape derivatives.

Let D and Ω be bounded domains in $\mathbb{R}^2$ such that $\overline{D} \subset \Omega$ so that D is the core and $\Omega \setminus D$ is the shell (coating). The conductivity distribution is given by

$$\sigma = \sigma_c \chi(D) + \sigma_s \chi(\Omega \setminus D) + \sigma_m \chi(\mathbb{R}^2 \setminus \Omega). \tag{1.3}$$

Here the conductivities σ_c, σ_s, and σ_m are assumed to be isotropic and homogeneous.

We design an optimization algorithm minimizing a cost functional given by

$$J[\sigma] := \frac{1}{2} \left| \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(\sigma) \right|^2,$$

for all a_α, b_β such that $\sum_\alpha a_\alpha x^\alpha$ and $\sum_\beta b_\beta x^\beta$ are harmonic polynomials of degree less than N (a given integer). We will call (a_α) and (b_β) harmonic coefficients and $\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}$ a harmonic combination of GPTs.

In order to compute the shape derivative of this functional, we derive an asymptotic expansion of the perturbations of the GPTs due to a small deformation of the boundary $\partial\Omega$. The derivation is rigorous and based on layer potential techniques in the same spirit as in [1]. It turns out that the shape derivative of $\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(\sigma)$ has a simple form. This is the