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FAST ALGORITHMS FOR HIGHER-ORDER SINGULAR VALUE DECOMPOSITION FROM INCOMPLETE DATA*

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Abstract

Higher-order singular value decomposition (HOSVD) is an efficient way for data reduction and also eliciting intrinsic structure of multi-dimensional array data. It has been used in many applications, and some of them involve incomplete data. To obtain HOSVD of the data with missing values, one can first impute the missing entries through a certain tensor completion method and then perform HOSVD to the reconstructed data. However, the two-step procedure can be inefficient and does not make reliable decomposition.

In this paper, we formulate an incomplete HOSVD problem and combine the two steps into solving a single optimization problem, which simultaneously achieves imputation of missing values and also tensor decomposition. We also present one algorithm for solving the problem based on block coordinate update (BCU). Global convergence of the algorithm is shown under mild assumptions and implies that of the popular higher-order orthogonality iteration (HOOI) method, and thus we, for the first time, give global convergence of HOOI.

In addition, we compare the proposed method to state-of-the-art ones for solving incomplete HOSVD and also low-rank tensor completion problems and demonstrate the superior performance of our method over other compared ones. Furthermore, we apply it to face recognition and MRI image reconstruction to show its practical performance.

Mathematics subject classification: 65F99, 9008, 90C06, 90C26.
Key words: multilinear data analysis, higher-order singular value decomposition (HOSVD), low-rank tensor completion, non-convex optimization, higher-order orthogonality iteration (HOOI), global convergence.

1. Introduction

Multi-dimensional arrays (or called *tensor*s) appear in many applications that collect data along multiple dimensions, including space, time, and spectrum, from different subjects (e.g., patients), and under different conditions (e.g., view points, illuminations, expressions). Higher-order singular value decomposition (HOSVD) [6] is an efficient way for dimensionality reduction and eliciting the intrinsic structure of the multi-dimensional data. It generalizes the matrix SVD and decomposes a multi-dimensional array into the product of a core tensor and a few orthogonal matrices, each of which captures the subspace information corresponding to one dimension (also called *mode* or *way*). The decomposition can be used for classification tasks including face recognition [41], handwritten digit classification [34], human motion analysis and recognition [40], and so on. HOSVD can also be applied to predicting unknown values while the acquired data is incomplete such as seismic data reconstruction [18] and personalized web search [37]. On imputing missing values, data fitting is the main goal instead of decomposition itself.

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However, there are applications that involve missing values and also require the decomposition such as face recognition [12], facial age estimation [11], and DNA microarray data analysis [28].

In this paper, we aim at finding an approximate HOSVD of a given multi-dimensional array with missing values. More precisely, given partial entries of a tensor $\boldsymbol{\mathcal{M}} \in \mathbb{R}^{m_1\times\ldots\times m_N}$, we estimate its HOSVD as $\boldsymbol{\mathcal{C}} \times_1 \mathbf{A}_1 \ldots \times_N \mathbf{A}_N$ such that the product is close to the underlying tensor $\boldsymbol{\mathcal{M}}$ and $\mathbf{A}_n$ can capture dominant subspace of the n-th mode of $\boldsymbol{\mathcal{M}}$ for all n, where $\boldsymbol{\mathcal{C}} \in \mathbb{R}^{r_1\times\ldots\times r_N}$ is a core tensor, $\mathbf{A}_n \in \mathbb{R}^{m_n\times r_n}$ has orthonormal columns for all n, and $\times_n$ denotes the mode-n tensor-matrix multiplication (see (1.2) below). To achieve the goal, we propose to solve the following *incomplete HOSVD* problem:

$$\begin{aligned} &\min_{\boldsymbol{\mathcal{C}},\mathbf{A}} \tfrac{1}{2}\|\mathcal{P}_\Omega(\boldsymbol{\mathcal{C}} \times_1 \mathbf{A}_1 \ldots \times_N \mathbf{A}_N - \boldsymbol{\mathcal{M}})\|_F^2, \\ &\text{s.t. } \mathbf{A}_n^\top \mathbf{A}_n = \mathbf{I},\ \mathbf{A}_n \in \mathbb{R}^{m_n\times r_n},\ n = 1,\ldots,N, \end{aligned} \tag{1.1}$$

where $\mathbf{A} = (\mathbf{A}_1,\ldots,\mathbf{A}_N)$, $(r_1,\ldots,r_N)$ is a given multilinear rank, $\mathbf{I}$ is the identity matrix of appropriate size, Ω indexes the observed entries, and $\mathcal{P}_\Omega$ is a projection that keeps the entries in Ω and zeros out others. Since only partial entries of $\boldsymbol{\mathcal{M}}$ are assumed known in (1.1), we cannot have $r_n = m_n$, $n = 1,\ldots,N$, because otherwise, it will cause overfitting problem. Hence, in general, instead of a full HOSVD, we can only get a truncated HOSVD of $\boldsymbol{\mathcal{M}}$ from its partial entries.

To get an approximate HOSVD of $\boldsymbol{\mathcal{M}}$ from its partial entries, one can also first fill in the unobserved entries through a certain tensor completion method and then perform some iterative method to have a (truncated) HOSVD of the estimated tensor. The advantage of our method is that it combines the two steps into solving just one problem and is usually more efficient and accurate. In addition, upon solving (1.1), we can also estimate the unobserved entries of $\boldsymbol{\mathcal{M}}$ from $\boldsymbol{\mathcal{C}} \times_1 \mathbf{A}_1 \ldots \times_N \mathbf{A}_N$ and thus achieve the tensor completion as a byproduct.

We will write (1.1) into one equivalent problem and solve it by the block coordinate descent (BCD) method. Although the problem is non-convex, we will demonstrate that (1.1) solved by the simple BCD can perform better than state-of-the-art tensor completion methods on reconstructing (approximate) low-multilinear-rank tensors. In addition, it can produce more reliable factors and as a result give higher prediction accuracies on certain classification problems such as the face recognition problem.

1.1. Related work

We first review methods for matrix and tensor factorization with missing values and then existing works on low-rank tensor completion (LRTC).

Matrix and tensor factorization with missing values

The matrix SVD from incomplete data has been studied for decades; see [20,33] for example. It can be regarded as a special case of (1.1) by setting $N = 2$ and restricting $\boldsymbol{\mathcal{C}}$ to be a nonnegative diagonal matrix. Further removing the orthogonality constraint on $\mathbf{A}_n$'s and setting $\boldsymbol{\mathcal{C}}$ as the identity matrix, (1.1) reduces to the matrix factorization from incomplete data (e.g., see [9]). Existing methods for achieving matrix SVD or factorization with missing values are mainly BCD-type ones such as the expectation maximization (EM) in [36] that alternates between imputation of the missing values and SVD computation of the most recently estimated matrix, and the successive over-relaxation (SOR) in [43] that iteratively updates the missing values and the basis and coefficient factors by alternating least squares with extrapolation. There are