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A CASCADIC MULTIGRID METHOD FOR EIGENVALUE PROBLEM*

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Abstract

A cascadic multigrid method is proposed for eigenvalue problems based on the multilevel correction scheme. With this new scheme, an eigenvalue problem on the finest space can be solved by linear smoothing steps on a series of multilevel finite element spaces and nonlinear correcting steps on special coarsest spaces. Once the sequence of finite element spaces and the number of smoothing steps are appropriately chosen, the optimal convergence rate with the optimal computational work can be obtained. Some numerical experiments are presented to validate our theoretical analysis.

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Key words: Eigenvalue problem, Cascadic multigrid, Multilevel correction scheme, Finite element method.

1. Introduction

The cascadic multigrid method proposed by [4,6] and analyzed by [17] is based on a hierarchy of nested meshes. Going from the coarsest level to the finest one, in each level, the discrete approximation obtained from the previous level acts as the starting value of a simple iterative solver (a smoother) like conjugate gradient. It is well known that for some certain linear systems (e.g., descretized by finite element method), a smoother can not eliminate the error effectively, and the part of error hard to be reduced is called *algebraic error*, which has been motivating the research on multigrid method. Therefore, to achieve the desired accuracy, the algebraic error on each level must be small enough. In cascadic multigrid method, this is achieved by increasing the number of smoothing iteration steps on coarser levels. Fortunately, the smaller dimensions of the problems on the coarser levels lead to the optimality of the complete algorithm. Requiring the number of operations which is proportional to the number of unknowns on the finest level, the algebraic error of the final approximation solution is of the same order as the discretization error of the finite element method. For more information about the cascadic multigrid method, please refer to [4, 6, 11, 17, 18, 20] and the references cited therein.

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In modern science and engineer, eigenvalue problems appear in many fields such as Physics, Chemistry, mechanics and material sciences. Recently, a type of multilevel correction method is proposed to solve eigenvalue problems in [13, 22]. In this multilevel correction scheme, the solution of eigenvalue problem on the final level mesh can be reduced to a series of solutions of boundary value problems on the multilevel meshes and a series of solutions of the eigenvalue problem on the coarsest mesh. Then it is natural to use the efficient linear solvers such as multigrid method and algebraic multigrid method to design the corresponding efficient eigenvalue solvers. It is well known that the cascadic multigrid method is simple and easy to be implemented. Therefore, the aim of this paper is to construct a cascadic multigrid method to solve the eigenvalue problem by transforming the eigenvalue problem solving to a series of smoothing iteration steps on the sequence of meshes and eigenvalue problem solving on the coarsest mesh by the multilevel correction method. Similarly to the cascadic multigrid for the boundary value problem, we also only do the smoothing steps for a boundary value problem by using the previous eigenpair approximation as the start value. As same as the cascadic multigrid method for boundary value problems, the numbers of smoothing iteration steps need to be increased in the coarse levels. The final eigenpair approximation has the same order algebraic error as the discretization error of the finite element method by organizing the suitable number of smoothing iteration steps on different levels. The original version of this paper is [9]. After that, there also have appeared a different cascadic multigrid method in [19] which is based on the shifted-inverse power iteration [8, 10, 16].

The rest of this paper is organized as follows. In the next section, we introduce the finite element method for the eigenvalue problem and the corresponding error estimates. A cascadic multigrid method for eigenvalue problem based on the multilevel correction scheme is presented and analyzed in Section 3. In Section 4, three numerical examples are presented to validate our theoretical analysis. Some concluding remarks are given in the last section.

2. Finite Element Method for Eigenvalue Problem

This section is devoted to introducing some notation and the finite element method for the eigenvalue problem. In this paper, we shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and semi-norms ([1]). For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v|_{\partial\Omega} = 0\}$, where $v|_{\Omega} = 0$ is in the sense of trace, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$. In some places, $\|\cdot\|_{s,2,\Omega}$ should be viewed as piecewise defined if it is necessary. The letter C (with or without subscripts) denotes a generic positive constant independent of mesh size which may be different at its different occurrences through the paper.

For simplicity, we consider the following model problem to illustrate the main idea: Find (λ, u) such that

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u) = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\mathcal{A}$ is a symmetric and positive definite matrix with suitable regularity, $\Omega \subset \mathcal{R}^d (d = 2, 3)$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and ∇, $\nabla\cdot$ denote the gradient, divergence operators, respectively.

In order to use the finite element method to solve the eigenvalue problem (2.1), we need to define the corresponding variational form as follows: Find $(\lambda, u) \in \mathcal{R} \times V$ such that $b(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v), \quad \forall v \in V, \tag{2.2}$$