Journal of Computational Mathematics http://www.global-sci.org/jcm
Vol.34, No.5, 2016, 511–531. doi:10.4208/jcm.1603-m2015-0317

OPTIMAL A POSTERIORI ERROR ESTIMATES OF THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS IN ONE SPACE DIMENSION*

Mahboub Baccouch
Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182
Email: mbaccouch@unomaha.edu

Abstract

In this paper, we derive optimal order *a posteriori* error estimates for the local discontinuous Galerkin (LDG) method for linear convection-diffusion problems in one space dimension. One of the key ingredients in our analysis is the recent optimal superconvergence result in [Y. Yang and C.-W. Shu, *J. Comp. Math.*, 33 (2015), pp. 323-340]. We first prove that the LDG solution and its spatial derivative, respectively, converge in the L^2-norm to $(p+1)$-degree right and left Radau interpolating polynomials under mesh refinement. The order of convergence is proved to be $p+2$, when piecewise polynomials of degree at most p are used. These results are used to show that the leading error terms on each element for the solution and its derivative are proportional to $(p+1)$-degree right and left Radau polynomials. We further prove that, for smooth solutions, the *a posteriori* LDG error estimates, which were constructed by the author in an earlier paper, converge, at a fixed time, to the true spatial errors in the L^2-norm at $\mathcal{O}(h^{p+2})$ rate. Finally, we prove that the global effectivity indices in the L^2-norm converge to unity at $\mathcal{O}(h)$ rate. These results improve upon our previously published work in which the order of convergence for the *a posteriori* error estimates and the global effectivity index are proved to be $p+3/2$ and $1/2$, respectively. Our proofs are valid for arbitrary regular meshes using P^p polynomials with $p \geq 1$. Several numerical experiments are performed to validate the theoretical results.

Mathematics subject classification: 65M15, 65M60, 65M50, 65N30, 65N50.
Key words: Local discontinuous Galerkin method, Convection-diffusion problems, Superconvergence, Radau polynomials, A posteriori error estimation.

1. Introduction

In this paper, we analyze a residual-based *a posteriori* error estimates of the spatial errors for the semi-discrete local discontinuous Galerkin (LDG) method applied to the following one-dimensional linear convection-diffusion equation

$$u_t + cu_x - ku_{xx} = f(x,t), \quad x \in [a,b],\ t \in [0,T], \tag{1.1a}$$

subject to the initial and periodic boundary conditions

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.1b}$$
$$u(a,t) = u(b,t), \quad u_x(a,t) = u_x(b,t), \quad t \in [0,T], \tag{1.1c}$$

where $c \geq 0$ is assumed to be a constant and $k > 0$ is the diffusion constant. For the sake of simplicity, we only consider the case of periodic boundary conditions. However, this assumption

* Received June 23, 2015 / Revised version received December 28, 2015 / Accepted March 11, 2016 / Published online September 14, 2016 /

is not essential. We note that if other boundary conditions (*e.g.*, Dirichlet or Neumann or mixed boundary conditions) are chosen, the LDG method can be easily designed; see [18, 39] for some discussion. In our analysis, the initial condition $u_0(x)$ and the source $f(x,t)$ are assumed to be sufficiently smooth functions with respect to all arguments so that the exact solution, $u(x,t)$, is a smooth function on $[a,b] \times [0,T]$.

The LDG method we discuss in this paper is an extension of the discontinuous Galerkin (DG) method aimed at solving differential equations containing higher than first-order spatial derivatives. The DG method is a class of finite element methods, using discontinuous, piecewise polynomials as the numerical solution and the test functions. It was first developed by Reed and Hill [35] for solving hyperbolic conservation laws containing only first order spatial derivatives in 1973. Consult [24] and the references cited therein for a detailed discussion of the history of DG methods and a list of important citations on the DG method and its applications. The LDG method for solving convection-diffusion problems was first introduced by Cockburn and Shu in [26]. Since then, LDG schemes have been successfully applied to hyperbolic, elliptic, and parabolic partial differential equations [2–4, 6, 7, 14, 16–18, 25–28, 33, 34, 37, 38], to mention a few. A review of the LDG methods is given in [10, 15, 16, 22–24].

There are many motivations for using LDG methods. With their carefully devised numerical fluxes, LDG methods are robust and high-order accurate, can achieve stability without slope limiters, and are locally (elementwise) mass-conservative. This last property is very useful in the area of computational fluid dynamics, especially in situations where there are shocks, steep gradients or boundary layers. Moreover, LDG methods are extremely flexible in the mesh-design; they can easily handle meshes with hanging nodes, elements of various types and shapes, and local spaces of different orders. They further exhibit strong superconvergence that can be used to estimate the discretization errors.

Recent work on the LDG method method for diffusion and convection-diffusion problems has been reviewed in [13, 39]. In particular, Cheng and Shu [18] studied the superconvergence property of the LDG method for linear hyperbolic and convection-diffusion equations in one space dimension. They proved superconvergence towards a particular projection of the exact solution. The order of superconvergence is proved to be $p + 3/2$, when p-degree piecewise polynomials with $p \geq 1$ are used. However, the superconvergence rate obtained in [18] is not optimal. In [7], we constructed and analyzed the global convergence of an implicit residual-based *a posteriori* error estimates. We applied the superconvergence results of Cheng and Shu [18] and proved that these estimates, at a fixed time t, converge to the true spatial error in the L^2-norm under mesh refinement. The order of convergence is proved to be $p + 3/2$. We further proved that the global effectivity indices converge to unity at $\mathcal{O}(h^{1/2})$ rate. In this paper, we improve upon the results in [7]. Recent optimal superconvergence results are used to obtain optimal convergence rate in the L^2-norm for the *a posteriori* error estimates and higher convergence rate for the global effectivity indices.

Recently, Yang and Shu [39] studied the superconvergence of the error for the LDG finite element method for one-dimensional linear parabolic equations when the alternating flux is used. They proved that the error between the LDG solution and the exact solution is $(p+2)$-th order superconvergent at the Radau points with suitable initial discretization. They also proved superconvergence towards a particular projection of the exact solution. The order of superconvergence is proved to be $p + 2$. Their analysis is valid for arbitrary regular meshes and for P^p polynomials with arbitrary $p \geq 1$. They performed numerical experiments to demonstrate that the superconvergence rates are optimal. More recently, Cao and Zhang [13] studied the