Journal of Computational Mathematics
Vol.34, No.2, 2016, 174–185.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1511-m2015-0297

GENERALIZED AUGMENTED LAGRANGIAN-SOR ITERATION METHOD FOR SADDLE-POINT SYSTEMS ARISING FROM DISTRIBUTED CONTROL PROBLEMS*

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Abstract

In this paper, a generalized augmented Lagrangian-successive over-relaxation (GAL-SOR) iteration method is presented for solving saddle-point systems arising from distributed control problems. The convergence properties of the GAL-SOR method are studied in detail. Moreover, when $0 < \omega \leq 1$ and $Q = \frac{1}{\gamma}I$, the spectral properties for the preconditioned matrix are analyzed. Numerical experiments show that if the mass matrix from the distributed control problems is not easy to inverse and the regularization parameter β is very small, the GAL-SOR iteration method can work well.

Mathematics subject classification: 65F10, 65F50.
Key words: PDE-constraint optimization, Saddle-point matrices, Augmented Lagrangian method, Convergence, Preconditioning.

1. Introduction

Consider the distributed control problem which consists of a cost functional to be minimized subject to a partial differential equation problem posed on a domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$:

$$\begin{aligned} &\min_{u,\ f} \ \frac{1}{2}\|u-u_*\|_2^2+\beta\|f\|_2^2, \\ &\text{subject to} \ \begin{cases} -\nabla^2 u = f & \text{in } \Omega, \\ u = g_1 \ \text{ on } \ \Gamma_1 \ , \\ \frac{\partial u}{\partial n} = g_2 \ \text{ on } \ \Gamma_2, \end{cases} \end{aligned} \tag{1.1}$$

where Γ_1 and Γ_2 are the boundary of Ω, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$. $\beta \in \mathbb{R}^+$ is a regularization parameter. The function u_* represents the desired state. For recent references on this topic, see [1, 2, 17, 19, 26–28].

There are two approaches to obtain the solution of the PDE-constrained optimal problem (1.1), i.e., the discretize-then-optimize approach and the optimize-then-discretize approach. Following the discretize-then-optimize approach [1,28], the discretized form for the minimization problem (1.1) can be written as [1, 6, 12, 17]:

$$\begin{aligned} &\min_{u,f} \ \frac{1}{2}u^TMu - u^Tp + \|u_*\|_2^2 + \beta f^TMf, \\ &\text{subject to} \ \ Ku = Mf + d, \end{aligned} \tag{1.2}$$

* Received May 22, 2015 / Revised version received October 24, 2015 / Accepted November 4, 2015 / Published online March 6, 2016 /

where $M \in \mathbb{R}^{m\times m}$ is the mass matrix and usually is symmetric definite positive. $K \in \mathbb{R}^{m\times m}$ is the stiffness matrix (the discrete Laplacian), $d \in \mathbb{R}^m$ represents the boundary data, and $p \in \mathbb{R}^m$ is the Galerkin projection of the discrete state u_*. By applying the Lagrangian multiplier method to (1.2), it follows the linear system

$$\begin{bmatrix} 2\beta M & 0 & -M \\ 0 & M & K^T \\ -M & K & 0 \end{bmatrix} \begin{bmatrix} f \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ p \\ d \end{bmatrix}, \tag{1.3}$$

where v is a vector of Lagrange multiplier. See [17,19] for more details.

Denote

$$A = \begin{bmatrix} 2\beta M & 0 \\ 0 & M \end{bmatrix}, \quad B = [-M \quad K], \quad y = [f^T,\ u^T]^T \quad \text{and} \quad c = [0^T,\ p^T]^T,$$

then the linear system (1.3) can be rewritten as the following saddle-point system:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}, \tag{1.4}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite (SPD) with $n = 2m$, $B \in \mathbb{R}^{m\times n}$ is full rank in row.

Systems of the form (1.4) appear in many applications such as constrained optimization [24], and constrained least squares problems [5,16,22]. Many efficient methods have been proposed for solving the linear system (1.4), see [8,9,20] and references therein.

In [14], Golub, Wu and Yuan proposed an SOR-like method for solving the linear system (1.4). Later, some researchers generalized the SOR-like method. For example, Drvishi and Hessari proposed an SSOR-like method in [10]. Shao, Shen and Li [23] generalized the SOR-like method by introducing a new parameter. Guo, Li and Wu proposed a modified SOR-like method [15]. Bai, Parlett and Wang [4] introduced another generalized SOR method and gave the optimal iteration parameter, and so on.

When the parameters are chosen appropriately, the existing methods are efficient. However, it is usually difficult to determine the iteration parameters.

For the PDE-constrained optimal problem, the regularization parameter β is often very small. Therefore, the (1,1) block matrix of the coefficient matrix of (1.4) is ill-conditioned. Then we can use the augmented Lagrangian technique [7] to rewrite the original saddle point system into an equivalent system. The equivalent form of the system (1.4) is:

$$\begin{bmatrix} A + \gamma B^T B & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} c + \gamma B^T d \\ -d \end{bmatrix},$$

where $\gamma > 0$ is a Lagrangian parameter.

In the following of this paper, we will use a generalized augmented Lagrangian technique to obtain the following equivalent form first:

$$\mathcal{A}x \equiv \begin{bmatrix} A + B^T Q^{-1} B & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} c + B^T Q^{-1} d \\ -d \end{bmatrix} \equiv b, \tag{1.5}$$

where $Q \in \mathbb{R}^{m\times m}$ is a generalized Lagrangian parameter matrix and usually is symmetric positive definite. In practical computations, the matrix Q is chosen such that it is easy to