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THE EXACT RECOVERY OF SPARSE SIGNALS VIA ORTHOGONAL MATCHING PURSUIT*

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Abstract

This paper aims to investigate sufficient conditions for the recovery of sparse signals via the orthogonal matching pursuit (OMP) algorithm. In the noiseless case, we present a novel sufficient condition for the exact recovery of all k-sparse signals by the OMP algorithm, and demonstrate that this condition is sharp. In the noisy case, a sufficient condition for recovering the support of k-sparse signal is also presented. Generally, the computation for the restricted isometry constant (RIC) in these sufficient conditions is typically difficult, therefore we provide a new condition which is not only computable but also sufficient for the exact recovery of all k-sparse signals.

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Key words: Compressed sensing, Sparse signal recovery, Restricted orthogonality constant (ROC), Restricted isometry constant (RIC), Orthogonal matching pursuit (OMP).

1. Introduction

Recovery of a sparse signal based on a small number of linear measurements is a fundamental problem in compressed sensing [10]. We consider the following model:

$$y = \Phi\beta + \epsilon, \tag{1.1}$$

where $y \in \mathbb{R}^m$ is an observation vector, $\Phi \in \mathbb{R}^{m\times n}$ is a known sensing matrix and $\epsilon \in \mathbb{R}^m$ is the measurement error vector. Suppose $\Phi = (\phi_1, \phi_2, \dots, \phi_n)$ where ϕ_i denotes the ith column of Φ. Throughout this paper we assume that the columns of Φ are normalized, i.e., $\|\phi_i\|_2 = 1$ for $i = 1, 2, \dots, n$. The goal of compressed sensing is to reconstruct the unknown $\beta \in \mathbb{R}^n$ based on y and Φ.

One of the most commonly used frameworks for the recovery of sparse signals is the Mutual Coherence Property introduced by Donoho and Huo in [11]. For a vector $\beta = (\beta(1), \dots, \beta(n)) \in \mathbb{R}^n$, the support of β is defined as $\mathrm{supp}(\beta) = \{i : \beta(i) \neq 0\}$ and β is said to be k-sparse if $|\mathrm{supp}(\beta)| \leq k$. The mutual coherence is defined by [11].

Definition 1.1. (Mutual Coherence [11]) *The mutual coherence μ of a matrix Φ is defined as*

$$\mu := \max_{i\neq j} |\langle \phi_i, \phi_j\rangle|. \tag{1.2}$$

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The validity of the OMP algorithm was investigated by Tropp [18] and Cai and Xu [4] in the framework of Mutual Coherence. In the noiseless case, Tropp [18] showed that $\mu < \frac{1}{2k-1}$ is a sufficient condition for the exact recovery of a k-sparse signal β, and Cai and Xu [4] showed that this condition is in fact sharp. When the linear measurement is corrupted by noise, Cai and Wang [2] considered two types of bounded noise. One is ℓ_2 bounded noise, i.e., $\|\epsilon\|_2 \le \eta_1$, for some constant $\eta_1 > 0$. The other is ℓ_∞ bounded noise, i.e., $\|\Phi^T \epsilon\|_\infty \le \eta_2$, for some constant $\eta_2 > 0$. In the ℓ_2 bounded noise case, if the conditions

$$\mu < \frac{1}{2k-1} \quad \text{and} \quad |\beta(i)| > \frac{2\eta_1}{1-(2k-1)\mu} \quad \big(i \in \operatorname{supp}(\beta)\big)$$

are satisfied, then the support of the k-sparse signal β can be recovered exactly via OMP. In the ℓ_∞ bounded noise case, a similar result was given.

In the framework of restricted isometry property (RIP), the validity of the OMP algorithm was investigated by Mo et al. [14], Wu et al. [17] and Cheng et al. [8]. Their results were related to the restricted isometry constant (RIC), that is defined by [7].

Definition 1.2. *Let $\Phi \in \mathbb{R}^{m\times n}$ be a matrix, and let $1 \le k \le n$ be an integer. The restricted isometry constant (RIC) of order k is defined as the smallest non-negative number δ_k^Φ such that for all k-sparse vectors $\beta \in \mathbb{R}^n$,*

$$(1-\delta_k^\Phi)\|\beta\|_2^2 \le \|\Phi\beta\|_2^2 \le (1+\delta_k^\Phi)\|\beta\|_2^2.$$

In the noiseless case, Mo and Shen [14] showed that under the condition $\delta_{k+1}^\Phi < \frac{1}{1+\sqrt{k}}$, OMP can exactly recover the k-sparse signal. In the ℓ_2 bounded noisy case, Wu et al. [17] showed that the support of the k-sparse signal β can be recovered exactly via OMP under the conditions

$$\delta_{k+1}^\Phi < \frac{1}{1+\sqrt{k}} \quad \text{and} \quad |\beta(i)| > \frac{\left(\sqrt{1+\delta_{k+1}^\Phi}+1\right)\eta_1}{1-(\sqrt{k}+1)\delta_{k+1}^\Phi} \quad \big(i \in \operatorname{supp}(\beta)\big).$$

In the ℓ_∞ bounded noise case, a similar result was given.

In this paper, some sufficient conditions based on the restricted orthogonality constant (ROC) are given. The following definition can be seen, e.g., in [5,15].

Definition 1.3. *Let $\Phi \in \mathbb{R}^{m\times n}$ be a matrix, and let $1 \le k_1, k_2 \le n$ be two integers. The restricted orthogonality constant (ROC) of order (k_1,k_2) is defined as the smallest non-negative number θ_{k_1,k_2}^Φ such that*

$$|\langle \Phi\beta_1, \Phi\beta_2\rangle| \le \theta_{k_1,k_2}^\Phi \|\beta_1\|_2 \|\beta_2\|_2,$$

for all k_1-sparse vector β_1 and k_2-sparse vector β_2 with disjoint supports. We set

$$\theta_{k_1,0}^\Phi = \theta_{0,k_2}^\Phi = 0.$$

For a matrix Φ with normalized columns, the mutual coherence is a special case of the ROC, i.e., $\mu = \theta_{1,1}^\Phi$. Roughly speaking, the RIC δ_k^Φ and ROC θ_{k_1,k_2}^Φ measure how far subsets of cardinality k of columns of Φ are to an orthonormal system. It is obvious that δ_k^Φ and θ_{k_1,k_2}^Φ are increasing in each of their indices.

In this paper, we establish some more relaxed conditions for sparse signals recovery via OMP. We show that the condition

$$\delta_k^\Phi + \sqrt{k}\theta_{1,k}^\Phi < 1$$