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AN EXPONENTIAL WAVE INTEGRATOR PSEUDOSPECTRAL METHOD FOR THE SYMMETRIC REGULARIZED-LONG-WAVE EQUATION*

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Abstract

An efficient and accurate exponential wave integrator Fourier pseudospectral (EWI-FP) method is proposed and analyzed for solving the symmetric regularized-long-wave (SRLW) equation, which is used for modeling the weakly nonlinear ion acoustic and space-charge waves. The numerical method here is based on a Gautschi-type exponential wave integrator for temporal approximation and the Fourier pseudospectral method for spatial discretization. The scheme is fully explicit and efficient due to the fast Fourier transform. Numerical analysis of the proposed EWI-FP method is carried out and rigorous error estimates are established without CFL-type condition by means of the mathematical induction. The error bound shows that EWI-FP has second order accuracy in time and spectral accuracy in space. Numerical results are reported to confirm the theoretical studies and indicate that the error bound here is optimal.

Mathematics subject classification: 65M12, 65M15, 65M70.
Key words: Symmetric regularized-long-wave equation, Exponential wave integrator, Pseudospecral method, Error estimate, Explicit scheme, Large step size.

1. Introduction

The symmetric regularized long wave (SRLW) equation reads,

$$u_t + \rho_x - u_{xxt} + \frac{1}{2}(u^2)_x = 0, \tag{1.1a}$$

$$\rho_t + u_x = 0, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.1b}$$

$$u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}, \tag{1.1c}$$

where $u(x,t)$, $\rho(x,t)$ are two real-valued functions, and u_0, v_0 are the given initial data. The equation is widely used for modeling the weakly nonlinear ion acoustic and space-charge waves [14, 23, 25, 26], and was first derived by C. E. Seyler and D. L. Fenstermacher in 1984 in [26] when they were working on a weakly nonlinear analysis of the cold-electron plasma equations appropriate for a strongly magnetized nonrelativistic electron beam such that the fluid motion is constrained to one direction. By eliminating ρ from (1.1), the SRLW equation has an equivalent single equation form as

$$u_{tt} - u_{xx} - u_{xxtt} + \frac{1}{2}\left(u^2\right)_{xt} = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

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which clearly shows that the SRLW equation is a wave type equation and due to this form, (1.1) is usually referred in the literatures as an equation rather than a system. The SRLW equation is symmetric in spatial and temporal derivatives, and is formally very similar to the regularized long wave equation that describes shallow water waves and plasma drift waves [3,4]. In some special physics situations, a dissipative version of SRLW is also proposed and studied in literature such as [28] and the references therein based on the SRLW equation (1.1).

Theoretically, the SRLW equation has gained many attentions. The local and global well-posedness of the SRLW has been studied by B. Guo in [19] and L. Chen in [13], and has been well-established by C. B. Brango in 2012 in [11]. The theoretical results therein indicate that the solutions of the SRLW equation decay very fast to zero at the far field, i.e.

$$\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} \rho(x,t) = 0,$$

at a fixed $t > 0$. The SRLW equation (1.1) has various conservation laws [13,26], such as the *energy*

$$E(u,\rho) := \int_{-\infty}^{\infty} \left(u^2(x,t) + u_x^2(x,t) + \rho^2(x,t)\right) \mathrm{d}x \equiv E(u_0,\rho_0), \tag{1.2}$$

and the two time *invariants*

$$I(u) = \int_{-\infty}^{\infty} u(x,t)\mathrm{d}x \equiv I(u_0), \qquad I(\rho) = \int_{-\infty}^{\infty} \rho(x,t)\mathrm{d}x \equiv I(\rho_0). \tag{1.3}$$

The energy (1.2) indicates that the two components u and ρ in the SRLW equation stay in different energy spaces. The SRLW equation (1.1) has also been remarkably pointed out to admit the solitary wave solutions (or solitons) [13,26] as

$$u_S(x,t;v,x_0) = \frac{3(v^2-1)}{v}\mathrm{sech}^2\left(\sqrt{\frac{v^2-1}{4v^2}}(x-vt+x_0)\right), \tag{1.4a}$$

$$\rho_S(x,t;v,x_0) = \frac{3(v^2-1)}{v^2}\mathrm{sech}^2\left(\sqrt{\frac{v^2-1}{4v^2}}(x-vt+x_0)\right), \quad x\in\mathbb{R}, \quad t\geq 0, \tag{1.4b}$$

where $|v| > 1$ is the velocity of the solitons and $x_0 \in \mathbb{R}$ is a shift in space. The importance of solitons in both theoretical studies of nonlinear wave equations and applications in many physical areas is already well demonstrated in [1,2,15]. L. Chen established the stability theory of these solitary waves (1.4) for SRLW equation in [13]. Integrability of the SRLW equation has been investigated in [26], where SRLW equation has been proved to be a nonintegrable system. Since the nonintegrable systems do not have the inverse scattering theory which is known as the superposition for nonlinear equations [29], so the interactions of the solitary waves are inelastic [12] and the dynamics of the SRLW equation are rather complicated analytical issues. Thus numerical methods and simulations are very much needed for the studies of the SRLW system.

For the numerical aspects, many finite difference (FD) time domain methods have been proposed and analyzed in literature. T. Wang etc. considered some conservative FD schemes that conserve the energy and the invariants in a discrete level in [31, 32]. However, these conservative schemes are fully implicit and at each step a full nonlinear problem has to be solved very accurately which is quite time-consuming. To improve the efficiency, some semi-implicit FD methods are also proposed in [31] that make the scheme at each time level a linear tri-diagonal