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ADAPTIVE CHOICE OF THE REGULARIZATION PARAMETER IN NUMERICAL DIFFERENTIATION*

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Abstract

We investigate a novel adaptive choice rule of the Tikhonov regularization parameter in numerical differentiation which is a classic ill-posed problem. By assuming a general unknown Hölder type error estimate derived for numerical differentiation, we choose a regularization parameter in a geometric set providing a nearly optimal convergence rate with very limited *a-priori* information. Numerical simulation in image edge detection verifies reliability and efficiency of the new adaptive approach.

Mathematics subject classification: 65D25, 65J20.
Key words: Numerical differentiation, Tikhonov regularization, Edge detection, Adaptive regularization.

1. Introduction

Numerical differentiation, which is a classic ill-posed problem, has been well-discussed in inverse problems community. Mathematically one aims to solve the Volterra integral equation of the first kind

$$Au(t) = \int_0^t u(s)ds = y(t) - y(0) \tag{1.1}$$

assuming $u(t) \in L^2(0,1)$ and $t \in [0,1]$. We assume that the measurement data contains some noise such that

$$y^\delta = Au + \xi \tag{1.2}$$

where A is the Volterra integral operator defined in (1.1) and ξ is measurement noise. Usually we assume that the measurement data y^δ is discrete and $\|y^\delta - y\| \leq \delta$ with a noise level δ known as *a-priori* information.

Numerical differentiation owns a wide range of applications including image edge detection, solutions of Abel integral equations, inverse problems arising from mathematical and physical systems etc [2]. Among them, the image edge detection, which appears in the analysis especially in grayscale level images, reconstructs a sharp jump or discontinuous points of a known source function/graph. The process becomes complicated if the source function is non-smoothing. In this sense, several methods have been used to explore the possibility to detect the image edges such as Jensen-Shannon divergence [3], pretopological formalism [8] and polynomial fitting [1].

The inherent ill-posedness of numerical differentiation makes the process numerically instable. More precisely, small disturb in measurement data will lead to huge error in computed derivatives. Notice the avoidless noise in real measurements, regularization methods must be

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taken to penalize the influence of noisy propagation. One general regularization scheme for solving an ill-posed problem is the Tikhonov regularization which is widely implemented among the well-studied ones. The classic Tikhonov regularization seeks for a minimizer of the following functional

$$\min J(u), \quad J(u) := \|Au - y^\delta\|^2 + \alpha\|u\|^2.$$

The regularization parameter α plays an important role of balancing the approximate errors and shall be calibrated very carefully [5].

At the same time, to incorporate discrete measurements and numerically smoothen approximate derivatives, a modified Tikhonov-type regularization scheme has been proposed for numerical differentiation and well-analyzed in [4, 11, 12]. Suppose $\{x_j\} = \{\frac{j}{n}\}$ $j = 0, 1, \cdots, n$ be the sampling points and $y = y(x)$ be a continuous function defined on $[0, 1]$. A noisy value of $y(x)$ at point x_j is given by y_j^δ which satisfies$|y(x_j) - y_j^\delta| < \delta$, $j = 0, 1, \cdots, n$, where δ is a given noise level. The modified Tikhonov functional is presented as follows:

$$\begin{aligned} &\min \Phi(f), \\ &\Phi(f) := \frac{1}{n}\sum_{i=1}^{n-1}\Big(f(x_i) - y_i^\delta\Big)^2 + \alpha\|f''\|^2 \end{aligned} \tag{1.3}$$

for all $f \in H^2(0, 1)$ with $y^\delta(0) = y(0)$ and $y^\delta(1) = y(1)$. In current work, instead of analyzing the properties under L^2-norm on the reconstructed gradient f', we will mainly discuss the L_∞-norm towards f' within the setting of (1.3). To be more precise, the approximated derivatives are obtained by solving (1.3) which is computationally cheap. Nevertheless, the regularization parameter is chosen by considering the L_∞-norm fitting the discontinuous properties along the image edges. Similar ideas also appear recently in [6]. By proposing a novel adaptive choice rule, we will determine the regularization parameter α in (1.3) giving the nearly order-optimal convergence rate under L_∞-norm with very limited *a-priori* information.

The manuscript is organized as follows. In Section 2, we set up the variational reconstruction problem under the framework of Tikhonov regularization (1.3) and present the adaptive choice rule within L_∞-norm setting. In Section 3, we will illustrate some academic examples and real images which support the arguments in Section 2. we will present some conclusion remarks in Section 4.

2. Adaptive Choice of the Regularization Parameter

Compared with existing results, we will derive the parameter choice rule based on L_∞-norm analysis to consider the local property of edge variation. Before formal analysis, we will first present two assumptions which are based on the noisy propagation term $\|u_\alpha - u_\alpha^\delta\|_{L_\infty}$ and the approximation error term $\|u^\dagger - u_\alpha\|_{L_\infty}$. Here we denote $u^\dagger$ as the exact solution to $y = Au$; u_α as the derivative of the minimizer of (1.3) with exact measurement $y^\delta = y$; u_α^δ as the derivative of the minimizer of (1.3) with noisy data y^δ.

Assumption 2.1. ([9]) *Let φ be any non-decreasing index function satisfying*

$$\gamma\frac{\alpha}{\varphi(\alpha)} \le \inf_{\alpha\le\lambda\le\sigma}\frac{\lambda}{\varphi(\lambda)}, \qquad 0 < \alpha \le \sigma$$