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ANALYSIS OF SHARP SUPERCONVERGENCE OF LOCAL DISCONTINUOUS GALERKIN METHOD FOR ONE-DIMENSIONAL LINEAR PARABOLIC EQUATIONS*

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Abstract

In this paper, we study the superconvergence of the error for the local discontinuous Galerkin (LDG) finite element method for one-dimensional linear parabolic equations when the alternating flux is used. We prove that if we apply piecewise k-th degree polynomials, the error between the LDG solution and the exact solution is $(k+2)$-th order superconvergent at the Radau points with suitable initial discretization. Moreover, we also prove the LDG solution is $(k+2)$-th order superconvergent for the error to a particular projection of the exact solution. Even though we only consider periodic boundary condition, this boundary condition is not essential, since we do not use Fourier analysis. Our analysis is valid for arbitrary regular meshes and for $\mathcal{P}^k$ polynomials with arbitrary $k \geq 1$. We perform numerical experiments to demonstrate that the superconvergence rates proved in this paper are sharp.

Mathematics subject classification: 65M60, 65M15.
Key words: Superconvergence, Local discontinuous Galerkin method, Parabolic equation, Lnitial discretization, Error estimates, Radau points.

1. Introduction

In this paper, we apply local discontinuous Galerkin (LDG) method to one-dimensional linear parabolic equation

$$\begin{aligned} &u_t = u_{xx}, && (x,t) \in [0,2\pi]\times[0,T], \\ &u(x,0) = u_0(x), && x \in [0,2\pi], \end{aligned} \tag{1.1}$$

where the initial datum u_0 is assumed to be sufficiently smooth. For simplicity, we will consider periodic boundary condition $u(0,t) = u(2\pi,t)$. However, this assumption is not essential since the proof is not based on Fourier analysis. We use piecewise k-th degree polynomials to approximate the solution in each cell and prove that, under suitable initial discretization, the rate of convergence for the error between the LDG solution and the exact solution is of $(k+2)$-th order at the Radau points. Moreover, we also prove the $(k+2)$-th order superconvergence of the error between the LDG solution and a particular type of projection of the exact solution estimated in L^p-norm, for any $1 \leq p \leq \infty$.

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The DG method was first introduced in 1973 by Reed and Hill [25], in the framework of neutron linear transport. Later, the method was applied by Johnson and Pitkäranta to a scalar linear hyperbolic equation and the L^p-norm error estimate was proved [23]. Subsequently, Cockburn et al. developed Runge-Kutta discontinuous Galerkin (RKDG) methods for hyperbolic conservation laws in a series of papers [16–19]. In [20], Cockburn and Shu first introduced the LDG method to solve the convection-diffusion equation. Their idea was motivated by Bassi and Rebay [8], where the compressible Navier-Stokes equations were successfully solved.

The superconvergence properties have been analyzed intensively. In [2, 5], Adjerid et al. studied the ordinary differential equations and proved the $(k+2)$-th order superconvergence of the DG solutions at the downwind-biased Radau points. For hyperbolic equations, the superconvergence results have been investigated by several authors [6,7,9,10,12,24,26,27]. Especially, in [26], we obtained sharp superconvergence for linear hyperbolic equations by using the dual argument, and this gives us the motivation to the prove the sharp superconvergence for linear parabolic equations. For convection-diffusion problems, in [3,4], the authors used numerical experiments to demonstrate the superconvergence of LDG solution at the Radau points. In [11], the steady state solution was studied and the superconvergence of the numerical fluxes was proved. In [13], Cheng and Shu discussed the superconvergence property of the LDG scheme for heat equation by using piecewise linear approximations and uniform meshes. Subsequently, they proved the $(k+\frac{3}{2})$-th order superconvergence when using piecewise k-th degree polynomials with arbitrary k on arbitrary regular meshes in [14]. However, the convergence rate obtained in [14] is not sharp. Numerical tests demonstrated that the error of the DG solution towards a particular projection of the exact solution is $(k+2)$-th order accurate, even on highly nonuniform meshes. In [14], the framework to prove the superconvergence results does not rely on Fourier analysis. Recently, in [9, 10], the authors studied the sharp superconvergence of linear hyperbolic and parabolic equations. In this paper, we give another proof for the estimate of the error between the exact and numerical solutions at the Radau points for linear parabolic equations. Motivated by [26], we adopt the dual argument to obtain the sharp rate of superconvergence and improve upon the result in [14]. The proof works for arbitrary regular meshes and schemes of any order.

The organization of this paper is as follows. In Section 2, we introduce the LDG scheme and state the main theorem. In Section 3, we present some preliminaries, including the norms we use throughout the paper, Radau polynomials, some essential properties of the finite element spaces, LDG spatial discretization as well as the error equations. Section 4 is the main body of the paper where the main theorem is proved. Numerical evidences about the sharpness of the superconvergence estimates are given in Section 5. In Section 6, we present some concluding remarks and remarks on future work. Finally, the initial discretization and properties about the test functions are given in Appendices A and B, respectively.

2. LDG scheme and the main result

In this section, we construct the LDG scheme for the linear parabolic equation (1.1). First, we divide the computational domain $\Omega = [0, 2\pi]$ into N cells

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi,$$

and define

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$$