Journal of Computational Mathematics
Vol.32, No.2, 2014, 205–214.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1401-m3837

SUPERCONVERGENCE ANALYSIS OF THE STABLE CONFORMING RECTANGULAR MIXED FINITE ELEMENTS FOR THE LINEAR ELASTICITY PROBLEM*

Dongyang Shi
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: shi_dy@zzu.edu.cn
Minghao Li
School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, China
Email: lyminghao@126.com

Abstract

In this paper, we consider the linear elasticity problem based on the Hellinger-Reissner variational principle. An $\mathcal{O}(h^2)$ order superclose property for the stress and displacement and a global superconvergence result of the displacement are established by employing a Clément interpolation, an integral identity and appropriate postprocessing techniques.

Mathematics subject classification: 65N15, 65N30.
Key words: Elasticity, Supercloseness, Global superconvergence.

1. Introduction

1.1. Introduction

In this paper, we consider the mixed finite element (for short MFE) approximation of a stress-displacement system derived from the Hellinger-Reissner variational principle for the linear elasticity problem. As is known to all, the MFE methods require that the pair of finite element spaces satisfying the B-B condition. Although there are a number of well-known stable MFEs for the analogous problems involving vector fields and scalar fields [1], the combination of the symmetry and continuity conditions of the stress field is a substantial additional difficulty. On the other hand, a lot of efforts, dating back four decades, have been devoted to develop stable MFEs for the linear elasticity problem, but no stable MFE scheme with polynomial shape functions are yielded. Not until the year 2002, were there some development in this direction. In [2], a sufficient condition was given and then a family of stable MFEs were constructed with respect to arbitrary triangular meshes, with 24 stress and 6 displacement degrees of freedom for the lowest order element, and an optimal order error estimate was obtained. An analogous family of conforming MFEs based on rectangular meshes were proposed in [3], involving 45 stress and 12 displacement degrees of freedom for the lowest order element. Two nonconforming triangular elements were presented in [4] with 12 degrees of freedom for the stress and 3 degrees of freedom for the displacement.

Although many stable elements have been constructed for this problem, they involve too much degrees of freedom. Recently, some more simple elements have been developed. In [5], a group of nonconforming rectangular elements were introduced, with the convergence order of

* Received August 9, 2011 / Revised version received December 4, 2013 / Accepted January 15, 2014 / Published online March 31, 2014 /

$\mathcal{O}(h)$ in L^2-norm for both the stress and the displacement, and the simplest element employed 12 degrees of freedom for the stress and 4 for the displacement. In [6], a family of conforming rectangular MFEs were proposed. It is closely related to one of the elements in [5]. Actually, the same finite element space is used for the displacement, while the space used to approximate the stress space is an extension of [5]. The lowest order pair in this family, with 17 degrees of freedom for the stress and 4 for the displacement, results in a convergence rate of $\mathcal{O}(h^2)$ for the stress and $\mathcal{O}(h)$ for the displacement in L^2-norm, respectively. In [7], a new family of minimal, any space-dimensional, symmetric, nonconforming mixed finite elements were presented. In 1D, it is nothing else but the 1D Raviart-Thomas element, which is the only conforming element in this family. In 2D and higher dimensions, they are new elements but of the minimal degrees of freedom. The total degrees of freedom for per element are 2 plus 1 in 1D, 7 plus 2 in 2D, and 15 plus 3 in 3D, respectively. In [8], the elements used in [7] were extended to conforming elements by enriching the spaces for both the stress and displacement, and the number of total degrees of freedom for per element are 10 plus 4 in 2D, and 21 plus 6 in 3D respectively, which are the simplest conforming rectangular elements so far.

On the other hand, the superconvergence study of the finite element methods is one of the most active topics for a long time in theoretical analysis and practical computations, and many valuable results about conforming and nonconforming finite elements have been obtained for different problems [9–16], but no consideration on this aspect is known about the finite elements of [6]. In this paper, at the first attempt, we will have a try to fill this gap. We obtain the supercloseness property of $\mathcal{O}(h^2)$ order for the stress and displacement and the superconvergence result of $\mathcal{O}(h^2)$ order for the displacement in L^2-norm through a Clément interpolation, an integral identity and interpolation postprocessing techniques.

The rest of this paper is organized as follows. In next section, some notations and preliminaries are introduced and the weak coercivity is established by the V-elliptic property and the B-B condition. Then we present the construction of finite element spaces in section 3. The last section is devoted to derive the supercloseness and global superconvergence of the displacement field.

2. Notations and Preliminaries

In this part, firstly we introduce some special functional spaces and operators. Let $\Omega \subset \mathcal{R}^2$ be a bounded convex domain, and $p, v = (v^{[1]}, v^{[2]})$ and $\tau = (\tau_{ij})_{2\times 2}$ be a function, vector-valued field and symmetric tensor, respectively. We define the following notions:

$$\operatorname{grad} p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix}, \qquad \operatorname{div} \tau = \begin{pmatrix} \partial \tau_{11}/\partial x + \partial \tau_{12}/\partial y \\ \partial \tau_{21}/\partial x + \partial \tau_{22}/\partial y \end{pmatrix},$$

$$\operatorname{grad} v = \begin{pmatrix} \partial v^{[1]}/\partial x & \partial v^{[1]}/\partial y \\ \partial v^{[2]}/\partial x & \partial v^{[2]}/\partial y \end{pmatrix}, \qquad \epsilon(v) = \frac{1}{2}(\operatorname{grad} v + (\operatorname{grad})^T v).$$

Let $\mathbb{S}$ denote the space of symmetric tensors, equipped with the inner product

$$(\sigma, \tau) = \int_\Omega \sigma : \tau, \quad \text{where} \quad \sigma : \tau = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij}.$$

The space $H^k(\Omega, X)$ is defined as

$$H^k(\Omega, X) = \Big\{ v \in L^2(\Omega, X) \,|\, D^\alpha v \in L^2(\Omega, X), \forall\, |\alpha| \le k \Big\},$$