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TIME-EXTRAPOLATION ALGORITHM (TEA) FOR LINEAR PARABOLIC PROBLEMS *

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Abstract

The fast solutions of Crank-Nicolson scheme on quasi-uniform mesh for parabolic problems are discussed. First, to decrease regularity requirements of solutions, some new error estimates are proved. Second, we analyze the two characteristics of parabolic discrete scheme, and find that the efficiency of Multigrid Method (MG) is greatly reduced. Numerical experiments compare the efficiency of Direct Conjugate Gradient Method (DCG) and Extrapolation Cascadic Multigrid Method (EXCMG). Last, we propose a Time-Extrapolation Algorithm (TEA), which takes a linear combination of previous several level solutions as good initial values to accelerate the rate of convergence. Some typical extrapolation formulas are compared numerically. And we find that under certain accuracy requirement, the CG iteration count for the 3-order and 7-level extrapolation formula is about 1/3 of that of DCG's. Since the TEA algorithm is independent of the space dimension, it is still valid for quasi-uniform meshes. As only the finest grid is needed, the proposed method is regarded very effective for nonlinear parabolic problems.

Mathematics subject classification: 65B05, 65M06, 65M12, 65N22.
Key words: Parabolic problem, Crank-Nicolson scheme, Error estimates, Time-extrapolation algorithm, CG-iteration.

1. Introduction

In modern science and technique, the high-dimensional parabolic problems, such as high heat transmission, superconductor, semi-conductor, nuclear-fusion and so on, are more and more important. However their computation is still very difficult.

As we all know, for the linear systems derived from elliptic problems, solving by direct methods is very difficult when the number of unknowns is more than tens of thousands. Therefore various iterative methods have emerged, such as

- Conjugate Gradient Method (CG) is efficient for solving symmetric positive definite systems, but the efficiency of CG reduces significantly when the condition number of coefficient matrix is greater than 10^3. The use of precondition techniques may be appropriate to improve the convergence rate, but the computational complexity increases.

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- Multigrid Method (MG) was presented by Fedorenko (1961) and Brandt (1977) [1]. For the linear systems derived from elliptic problems, the computational work $W = O(N)$ of MG is proportional to the number of unknowns N. So MG, widely used in scientific and engineering computing, has become one of the most effective algorithms to solve large scale problem. It should be noted that grids generated by MG method just satisfy the requirements for superconvergence and extrapolation.

- Cascadic Multigrid Method (CMG) presented by Bornemann [2], Deuflhard [8] and Shaidurov [17] is an one-way multigrid method which may be viewed as a multilevel method without the coarse mesh correction. Since 1998, Shi et al. made a lot of theoretical analysis [19, 20]. Because of its high efficiency, CMG has been quickly applied to a series of problems [10, 15, 16, 19, 24].

In recent years, we have proposed an extrapolation cascadic multi-grid method (EXCMG) [3, 6] and a new extrapolation formula, which use a linear combination of the solutions on previous 2 level coarse meshes to provide a good initial value of finite element solution on next-level fine mesh. This method is of high accuracy and converges for both the function and its derivative. The numerical experiments show that the work of EXCMG is close to that of MG algorithm for simple linear problems. However, EXCMG requires less iterations on the finest mesh and converges more quickly, see [7] for details.

Of course, the above methods can be used to solve parabolic problems, direct methods or indirect methods (i.e. the combination and iteration on multi-levels in time). Brandt studied early the indirect method, Hackbush [12] also suggested a time parallel MG algorithm. Later, Horton [13](1992), Horton and Vandewalle [14](1995), Gander and Vandewalle [11](2007) presented further developments. Shi and Xu [19](2000), Du and Ming [10](2008) directly used the CMG to solve the resulting elliptic problems with a discrete in time formulation for parabolic problems. In a word, all these methods are effective.

However the linear systems derived from parabolic problems have the own characteristics. For example, the condition number, $Cond(A) \approx 4r, \quad r = a^2k/(2h^2)$, is much smaller than the condition number of corresponding elliptic problems ($O(h^{-2})$). And the solution U^{j-1} obtained by previous level provides a good initial value of the solution U^j of the present level ($U^j = U^{j-1} + O(k)$). Numerical experiments show that for elliptic problems MG and EXCMG have the absolute advantage, but for parabolic problems things change. Especially when the condition number is not large(such as $r < 1000$), the efficiency of the two algorithms is greatly reduced, and they have lost the absolute advantage even though still better than the direct CG-iteration. Therefore we should develop other efficient algorithms for parabolic problems.

C.C. Douglas [9](1996) predicted that "some excellent time-extrapolation methods exist which can be coupled to conjugate gradient- like methods. Once the first few time steps are solved (slowly), the extrapolation method provides such a good initial value to the solution on the next time step that only a few (say, one or two) iterations of the conjugate gradient-like method are needed before moving on. In this situation multigrid cannot compete." However the numerical experiments and our research show that this problem is quite complicated, far from being so simple as he said.

In this paper we should derive some new estimates of Crank-Nicolson scheme at first , then we will discuss the following algorithms used to solve parabolic problems:

- Direct CG-iteration (DCG) is acceptable when $r < 700$.