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NUMERICAL STUDIES OF A CLASS OF COMPOSITE PRECONDITIONERS*

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Abstract

In this paper, we study a composite preconditioner that combines the modified tangential frequency filtering decomposition with the ILU(0) factorization. Spectral property of the composite preconditioner is examined by the approach of Fourier analysis. We illustrate that condition number of the preconditioned matrix by the composite preconditioner is asymptotically bounded by $\mathcal{O}(h_p^{-\frac{2}{3}})$ on a standard model problem. Performance of the composite preconditioner is compared with other preconditioners on several problems arising from the discretization of PDEs with discontinuous coefficients. Numerical results show that performance of the proposed composite preconditioner is superior to other relative preconditioners.

Mathematics subject classification: 65F10, 65N22.
Key words: Preconditioner, ILU, Tangential frequency filtering decomposition, GMRES.

1. Introduction

We consider preconditioning techniques for solving systems of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1.1}$$

with

$$\mathbf{A} = \begin{bmatrix} D_1 & U_1 & & \\ L_1 & D_2 & \ddots & \\ & \ddots & \ddots & U_{n_x-1} \\ & & L_{n_x-1} & D_{n_x} \end{bmatrix} \in \mathcal{R}^{N\times N}, \quad b \in \mathcal{R}^N,$$

where n_x denotes the number of grid points in the x-direction, $D_i \in \mathcal{R}^{n_i\times n_i}$, $L_i \in \mathcal{R}^{n_{i+1}\times n_i}$, $U_i \in \mathcal{R}^{n_i\times n_{i+1}}$, and $N = \sum_{i=1}^{x} n_i$ is the total number of grid points. Such kinds of problems frequently arise from numerical solution of partial differential equations [31, 41, 42, 47]. Due to prohibitive memory requirement, direct methods are generally not acceptable, especially for 3D problems. In recent years, Krylov subspace methods combining with appropriate preconditioners have become the natural choice [6, 27, 37, 43, 47]. It is known that convergence rate of preconditioned Krylov subspace methods heavily depend on the spectrum distribution of the preconditioned matrix, and preconditioning plays a key role in making spectrum distribution

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avail for fast convergence [43,46]. Therefore, a great deal of effort has been put into the development of efficient preconditioners. The multigrid methods are well known for their efficiency of reducing the high-frequency and low-frequency components of the error by complementary schemes [52]. However, these approaches may become ineffective when dealing with general sparse systems of linear equations. Another type of efficient preconditioning techniques are constructed by making use of spectrum information of the preconditioned matrix [23,29,38,44,45]. They are able to get rid of the influence of eigenvalues close to zero, which is generally difficult to handle by conventional incomplete factorization preconditioners.

It is well known that a wide class of preconditioners are based on incomplete factorizations, e.g., BILU, ILUT, SOR, HSS [4,6,10,14,15,28,30,31,43]. In the present work, we mainly focus on a class of incomplete factorization $\mathbf{M}$ that enables a so called right filtering property

$$(\mathbf{M}-\mathbf{A})\mathbf{f}=0,$$

or left filtering property

$$\mathbf{f}^T(\mathbf{M}-\mathbf{A})=0,$$

for a vector $\mathbf{f}$. The idea of filtering is longstand technique, and it is utilized in a nested factorization [3] by J. R. Appleyard, and popularized in recent years by G. Wittum and his successors in [1,2,19,21,22,48–51]. In particular, Tangential Frequency Filtering Decomposition (TFFD) preconditioner proposed by Achdou and Nataf [2] is an efficient filtering preconditioner. It is constructed by using the eigenvector associated with the smallest eigenvalue as a filtering vector. As the ILU(0) preconditioner is efficient in reducing the influence of the higher part of the spectrum [24] and the TFFD is efficient in removing the influence of low part, it is suggested in [2] to combine the TFFD with the classical ILU(0) in a multiplicative way. The combination results in a composite preconditioner which is efficient on some challenging problems with highly discontinuous coefficients. The selection of the filtering vector is an important issue. The choice of the filtering vectors is investigated in [32], and several different kinds of filtering methods are compared in [32]. The results reveal that $\mathbf{e}^T=[1,\dots,1]^T$ is a reasonable choice for a wide range of problems. It has been illustrated that the use $\mathbf{e}$ as a filtering vector is robust, and can save the cost in forming the filtering vector [32].

Fourier analysis is a classical scheme for analyzing both differential equations and discrete solution methods for time dependent problems [20, 42]. It is popularized by T.F. Chan etc [24, 33, 40] for analyzing algebraic preconditioners and classical iterative methods. On some point-wise incomplete factorization type preconditioners, for example, ILU(0) [34], modified ILU (MILU) [30] and relaxed ILU (RILU) [5], Fourier analysis has been carried out in [24–26]. The Fourier analysis of block ILU and MILU factorization preconditioners is considered in [40] on a time-dependent hyperbolic PDE problem. In [2] the TFFD preconditioner is analyzed by the approach of Fourier method, and an optimal modification of TFFD preconditioner is derived. The preconditioner is called Modified Tangential Frequency Filtering Decomposition (MTFFD) preconditioner. It is illustrated that the condition number of the MTFFD preconditioned matrix is asymptotically bounded by $\mathcal{O}(h^{-\frac{2}{3}})$. Compared with the asymptotic bounds of the condition number by using some classical incomplete factorizations like ILU(0), BILU and MILU ($\mathcal{O}(h^{-2})$ or $\mathcal{O}(h^{-1})$) [4,25,30,40], we can see that the bound obtained by MTFFD is considerably better. Numerical tests on some discontinuous problems also illustrate that MTFFD can improve the performance of TFFD.

In this paper, we investigate a composite preconditioner, which is constructed by combing the MTFFD preconditioner with the ILU(0) preconditioner in a multiplicative way. The com-