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ON THE ERROR ESTIMATES OF A NEW OPERATE SPLITTING METHOD FOR THE NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, a new operator splitting scheme is introduced for the numerical solution of the incompressible Navier-Stokes equations. Under some mild regularity assumptions on the PDE solution, the stability of the scheme is presented, and error estimates for the velocity and the pressure of the proposed operator splitting scheme are given.

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1. Introduction

In this paper, we consider the numerical approximation of the unsteady Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \triangledown)\mathbf{u} + \triangledown p = \mathbf{f} \quad \text{in}\, \Omega \times [0,T], \\ \triangledown \cdot \mathbf{u} = 0 \quad \text{in}\, \Omega \times [0,T], \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^d with a sufficiently regular boundary $\partial\Omega$. $\mathbf{u}$, p are the velocity, pressure of the flow respectively, and $\nu = \frac{1}{Re}$ is the kinematic viscosity coefficient, Re is the Reynolds number.

For the well-posedness, the equations are supplemented with appropriate initial and boundary condition:

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 \quad \mathbf{x} \in \Omega\, , \mathbf{u}(\mathbf{x},t) = g(\mathbf{x},t) \quad (\mathbf{x},t) \in \partial\Omega \times [0,T]. \tag{2.2}$$

The difficulties for the numerical simulation of incompressible flows are mainly of two kinds: nonlinearity and incompressibility, the velocity and the pressure are coupled by the incompressibility constraint, which requires that the solution spaces satisfy the so called inf-sup condition. To overcome these difficulties, operator splitting methods and projection methods, which can be viewed as fractional step methods, are introduced. Fractional step methods allow to separate the effects of the different operators appearing in the equation by splitting each step into several sub-steps in order to reduce the cost of simulations.

The origin of this category of methods is due to the work of Chorin [1] and Temam [2], i.e., the so called projection method, in which the second step consists of the projection of

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an intermediate velocity field onto the space of solenoidal vector fields. The most attractive feature of projection methods is that, at each time step, one only needs to solve a sequence of decoupled elliptic equations for the velocity and the pressure, which makes the projection method very efficient for large scale numerical simulations. Guermond, Minev and Shen in [3] review theoretical and numerical convergence results available for projection methods. In[4-7], the analysis on first-order accurate schemes in time are presented. In [8,9], Shen derived a second-order error estimates for the projection method. However, several issues related to these methods still deserve further analysis, and perhaps the most important ones are the behavior of the computed pressure near boundaries and the stability of the pressure itself. The incompatibility of the projection boundary conditions may introduce a numerical boundary layer of size $O(\sqrt{\nu\Delta t})$ [10,11], where ν is the kinematic viscosity and Δt is the time step size. In addition, these methods have a main disadvantage that splitting error is inevitable unless the operators are commute.

In this paper, we will consider the non-stationary Navier-Stokes equations with Dirichlet boundary conditions and provide some error estimates for both velocity and pressure approximations by the operator splitting scheme. It is a two-step scheme, which allows to enforce the original boundary conditions of the problem in all substeps of the scheme[12].

The paper is organized as follows: In Section 2, we introduce some function and space notations and regularity assumptions for the PDE solution. In Section 3, we describe a new operator splitting method. In Section 4, the proof of the stability of the new method is given. In Section 5, we give an error analysis of this method. Error estimates for both velocity and pressure are obtained. Finally, numerical test results are presented in Section 6 to verify the theoretical results of Section 5.

2. Function Setting

In order to study approximation scheme for the problem (1.1). The following notations and assumptions are introduced.

we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^d$. The space $H_0^1(\Omega)$ and $H_0^1(\Omega)^d$ are equipped with their usual norm, i.e.,

$$\|\mathbf{u}\|_1^2 = \int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x}.$$

The norm in $H^s(\Omega)$ will be denoted simply by $\|\cdot\|_s$. We will use $<\cdot,\cdot>$ to denote the duality between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$ for all $s>0$.

The following subspace is also be introduced:

$$V = \Big\{\mathbf{u} \in H_0^1(\Omega)^d : \mathrm{div}\mathbf{u} = 0\Big\},$$
$$H = \Big\{\mathbf{u} \in L^2(\Omega)^d : \mathrm{div}\mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n} = 0)\Big\}.$$

For the treatment of the convective term, the following trilinear form is considered

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}d\mathbf{x}.$$

It is well known that $b(\cdot,\cdot,\cdot)$ is continuous in $H^{m_1}(\Omega)\times H^{m_2+1}(\Omega)\times H^{m_3}(\Omega)$, provided $m_1+m_2+m_3\geq d/2$ if $m_i\neq d/2,\ \ i=1,2,3$, and this form is skew-symmetric with respect to its