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OPTIMAL POINT-WISE ERROR ESTIMATE OF A COMPACT FINITE DIFFERENCE SCHEME FOR THE COUPLED NONLINEAR SCHRÖDINGER EQUATIONS*

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Abstract

In this paper, we analyze a compact finite difference scheme for computing a coupled nonlinear Schrödinger equation. The proposed scheme not only conserves the total mass and energy in the discrete level but also is decoupled and linearized in practical computation. Due to the difficulty caused by compact difference on the nonlinear term, it is very hard to obtain the optimal error estimate without any restriction on the grid ratio. In order to overcome the difficulty, we transform the compact difference scheme into a special and equivalent vector form, then use the energy method and some important lemmas to obtain the optimal convergent rate, without any restriction on the grid ratio, at the order of $O(h^4+\tau^2)$ in the discrete L^∞-norm with time step τ and mesh size h. Finally, numerical results are reported to test our theoretical results of the proposed scheme.

Mathematics subject classification: 65M06, 65M12.
Key words: Coupled nonlinear Schrödinger equations, Compact difference scheme, Conservation, Point-wise error estimate.

1. Introduction

In this paper, we consider the coupled nonlinear Schrödinger (CNLS) equations

$$i\partial_t u + k\partial_{xx} u + (|u|^2 + \beta|v|^2)u = 0, \quad x \in \Omega \subseteq \mathbb{R}, \quad t > 0, \tag{1.1}$$

$$i\partial_t v + k\partial_{xx} v + (|v|^2 + \beta|u|^2)v = 0, \quad x \in \Omega \subseteq \mathbb{R}, \quad t > 0, \tag{1.2}$$

which arise in a great variety of physical situations. In fiber communication system, such equations have been shown to govern pulse propagation along orthogonal polarization axes in nonlinear optical fibers and in wavelength-division-multiplexed systems [25,32,41]. Here $u(x,t)$ and $v(x,t)$ are unknown complex-valued wave functions, k describes the dispersion in the optic fiber, β is defined for birefringent optic fiber coupling parameter, Ω is a bounded computational domain, i is the imaginary unit, i.e. $i^2=-1$. These equations also model two-component Bose-Einstein condensation and beam propagation inside crystals, photorefractives as well as water wave interactions.

There are many studies on numerical studying of the CNLS equations. In [1, 2, 42], some efficient time-splitting spectral methods were given to study the dynamics of two-component Bose-Einstein condensate. In [30], a multi-symplectic method was constructed and the solitons collision was simulated. In [29], a nonlinear implicit conservative scheme was proposed for the

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strong coupling of Schrödinger equations and both the analytic and numerical solutions were discussed. In [15–17] a Crank-Nicolson difference scheme, a linearized implicit scheme and a compact difference scheme were presented and some numerical experiments were given. In [13], Ismail discretized the space derivative by central difference formulas of fourth-order, then solved the resulting ordinary differential system by the fourth-order explicit Runge-Kutta method. The linearly convergence of all of the difference schemes in [13,15–17] was proved by $von\ Neumann$ method. In [14], Galerkin finite element method was proposed to solve the CNLS equation. In [33], Wang discussed the splitting spectral method for solving the CNLS equation. In [35], Wang et al. proposed and studied a nonlinear symplectic difference scheme. They proved the existence, uniqueness and second order convergence in l^2-norm under some restrictions on the grid ratios, and proposed an iterative algorithm for solving the difference scheme. In [31], Sun and Zhao also studied the nonlinear difference scheme proposed in [16,28,29]. They proved the existence, uniqueness and second order convergence in l^∞ norm (the discrete L^∞ norm), and proposed another interesting iterative algorithm for solving the nonlinear scheme. In [36], the optimal error estimate in l^∞ norm of the linearized difference scheme proposed in [17,34] was established.

Recently, there has been growing interest in high-order compact methods for solving partial differential equations, see, e.g., [4–7,9–12,18–23,26,40,43]. It was shown that the high-order difference methods play an important role in the simulation of high frequency wave phenomena. However, due to the difficulty caused by the compact difference on the nonlinear term, the energy method can not be used directly on the compact difference scheme, and so there is few proof of the unconditional error estimate in the l^∞-norm of any a compact difference scheme for nonlinear partial differential equations. In [37–39], without any restrictions on the grid ratios, we established the optimal l^∞-error estimates of some compact difference schemes for the nonlinear Schrödinger equation (NLSE) with *periodic* boundary conditions where the circulant coefficient matrix was used, but the technique can not be extended directly to NLSE with Dirichlet boundary condition because the coefficient matrix is no longer circulant.

In this paper, we introduce an efficient compact difference scheme for the CNLS equation on a finite domain $\Omega = [L_1, L_2]$ with initial conditions

$$u(x,0) = \psi(x), \quad v(x,0) = \phi(x), \quad x \in [L_1, L_2], \tag{1.3}$$

and homogeneous Dirichlet boundary conditions

$$u(L_1,t) = u(L_2,t) = 0, \quad v(L_1,t) = v(L_2,t) = 0, \quad t > 0, \tag{1.4}$$

where $\psi(x)$ and $\phi(x)$ are prescribed smooth functions vanishing at points $x = L_1$ and $x = L_2$.

The problem (1.1)-(1.4) has two kinds of standard conserved quantities, i.e., the total masses

$$M_1(t) := \int_{L_1}^{L_2} |u(x,t)|^2 dx \equiv M_1(0), \quad t \geq 0, \tag{1.5}$$

$$M_2(t) := \int_{L_1}^{L_2} |v(x,t)|^2 dx \equiv M_2(0), \quad t \geq 0, \tag{1.6}$$

and energy

$$\begin{aligned} E(t) := \frac{k}{2} \int_{L_1}^{L_2} \Big(|\partial_x u(x,t)|^2 + |\partial_x v(x,t)|^2\Big) dx - \frac{1}{4} \int_{L_1}^{L_2} \Big(|u(x,t)|^4 + |v(x,t)|^4\Big) dx \\ -\frac{\beta}{2} \int_{L_1}^{L_2} |u(x,t)|^2 |v(x,t)|^2 dx \equiv E(0), \quad t \geq 0. \end{aligned} \tag{1.7}$$