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A NEW DIRECT DISCONTINUOUS GALERKIN METHOD WITH SYMMETRIC STRUCTURE FOR NONLINEAR DIFFUSION EQUATIONS*

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Abstract

In this paper we continue the study of discontinuous Galerkin finite element methods for nonlinear diffusion equations following *the direct discontinuous Galerkin (DDG) methods for diffusion problems* [17] and *the direct discontinuous Galerkin (DDG) methods for diffusion with interface corrections* [18]. We introduce a numerical flux for the test function, and obtain a new direct discontinuous Galerkin method with symmetric structure. Second order derivative jump terms are included in the numerical flux formula and explicit guidelines for choosing the numerical flux are given. The constructed scheme has a symmetric property and an optimal $L^2(L^2)$ error estimate is obtained. Numerical examples are carried out to demonstrate the optimal $(k+1)th$ order of accuracy for the method with P^k polynomial approximations for both linear and nonlinear problems, under one-dimensional and two-dimensional settings.

Mathematics subject classification: 65M60.
Key words: Discontinuous Galerkin Finite Element method, Diffusion equation, Stability, Convergence.

1. Introduction

This paper is a continuous study following [17] and [18] regarding a discontinuous Galerkin finite element method for solving time dependent nonlinear diffusion equations of the form

$$U_t - \nabla \cdot (A(U)\nabla U) = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, the matrix $A(U) = (a_{ij}(U))$ is symmetric and positive definite, and U is an unknown function of $(\mathbf{x}, t)$ with $\mathbf{x} \in \Omega$.

The discontinuous Galerkin (DG) method is a finite element method with discontinuous piecewise function space for the numerical solution and the test functions. Lacking the restrictions of continuities across the computational cells makes these methods extremely flexible. As a result, DG methods have found application in diverse areas. The application of DG methods to hyperbolic problems has been quite successful since it was originally introduced by Reed and Hill [21] in 1973 for neutron transport equations. A major development of the DG method for nonlinear hyperbolic conservation laws was carried out by Cockburn, Shu, etc. We refer to [9–11] for reviews and further references of DG methods for hyperbolic type of PDEs.

However, application of DG method to diffusion problems has been a challenging task because of the subtle difficulty in defining appropriate numerical fluxes for diffusion terms, see

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e.g. [23]. There have been several DG methods suggested in literature for solving diffusion problems. One class is the *interior penalty* (IP) method, which dates back to 1982 as the symmetric interior penalty (SIPG) method by Arnold [1] (also by Baker in [3] and Wheeler in [28]). We also have the Baumann and Oden method [5, 20], the NIPG method [22] and the IIPG method [13]. Another class is the local discontinuous Galerkin (LDG) methods introduced in [12] by Cockburn and Shu (originally proposed by Bassi and Rebay [4] for compressible Navier-Stokes equations). We refer to the unified analysis article [2] in 2002 for different DG methods involving diffusion term and background references for other DG methods. More recent works include those by Van Leer and Nomura in [26], Gassner et al. in [16], Cheng and Shu in [8] and Brenner et al. in [6].

Recently in [17], we develop a direct discontinuous Galerkin (DDG) method for diffusion equations. The scheme is based on the direct weak formulation of (1.1), and a general numerical flux formula for the solution derivative is proposed. An optimal kth order error estimate in an energy norm is obtained with P^k polynomial approximations for linear diffusion equations. However, numerical experiments in [17] show that when measured under L^2 and L^∞ norms, the scheme accuracy is sensitive to the coefficients in the numerical flux formula. That is, for higher order P^k ($k \geq 4$) polynomial approximations it is difficult to identify suitable coefficients in the numerical flux to obtain optimal $(k+1)$th order of accuracy. In [18], extra interface correction terms are introduced into the scheme formulation, and a refined version of the DDG method is obtained. A simpler numerical flux formula is used in [18] and numerically optimal $(k+1)th$ order of accuracy is achieved for any P^k polynomial approximations.

The DDG method [17] and the DDG method with interface corrections [18] are schemes which both lack symmetric properties. Thus it is difficult to obtain L^2 error analysis. In this work, we introduce a numerical flux for the test function derivative and include more interface terms in the scheme formulation. With the same numerical flux formula for the solution derivative and test function derivative, the bilinear form for the diffusion term thus obtained has a symmetric property. This symmetric structure is the key to further prove an optimal $L^2(L^2)$ error estimate for the DG solution. Also, new guidelines for choosing admissible numerical fluxes are given. The symmetric DDG method is not sensitive to the coefficients in the numerical flux formula. There exists a large class of admissible numerical fluxes that lead to the optimal convergence. Compared to the SIPG method [1], the penalty coefficient estimate can be decreased from k^2 to $k^2/4$. One-dimensional and two-dimensional numerical examples are carried out and we obtain $(k+1)th$ optimal order of accuracy with piecewise P^k polynomial approximations for both linear and nonlinear diffusion problems.

In this paper we use uppercase letters to represent the exact solution and lowercase letters to represent the DG numerical solution and test functions. The rest of the paper is organized as follows. In §2, we describe the scheme formulation for the linear and nonlinear one-dimensional diffusion equations, present admissibility and stability results, and establish an energy norm error estimate for the linear case. In §3, extension to two-dimensional diffusion problems is given. The optimal $L^2(L^2)$ error estimate for the linear two-dimensional equation is presented in §4. Finally numerical examples are shown in §5.

2. One-Dimensional Diffusion Equations

2.1. Scheme formulation for 1-D linear diffusion equation

In this section, we present the new discontinuous Galerkin method with the following 1-D