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ON THE QUASI-RANDOM CHOICE METHOD FOR THE LIOUVILLE EQUATION OF GEOMETRICAL OPTICS WITH DISCONTINUOUS WAVE SPEED*

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Abstract

We study the quasi-random choice method (QRCM) for the Liouville equation of geometrical optics with *discontinuous* local wave speed. This equation arises in the phase space computation of high frequency waves through interfaces, where waves undergo partial transmissions and reflections. The numerical challenges include interface, contact discontinuities, and measure-valued solutions. The so-called QRCM is a random choice method based on quasi-random sampling (a deterministic alternative to random sampling). The method not only is viscosity-free but also provides faster convergence rate. Therefore, it is appealing for the problem under study which is indeed a Hamiltonian flow. Our analysis and computational results show that the QRCM 1) is almost first-order accurate even with the aforementioned discontinuities; 2) gives sharp resolutions for all discontinuities encountered in the problem; and 3) for measure-valued solutions, does not need the level set decomposition for finite difference/volume methods with numerical viscosities.

Mathematics subject classification: 35L45, 65M06, 11K45.
Key words: Liouville equation, High frequency wave, Interface, Measure-valued solution, Random choice method, Quasi-random sequence.

1. Introduction

In this paper, we study a type of Monte Carlo methods for numerical computation of the Liouville equation in geometrical optics. Let $f(\mathbf{x},\mathbf{v},t)$ be the energy density distribution of waves that depends on position $\mathbf{x}$, slowness vector $\mathbf{v}$, and time t, then the Liouville equation reads

$$f_t + \nabla_{\mathbf{v}} H \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} H \cdot \nabla_{\mathbf{v}} f = 0, \quad t>0,\ \mathbf{x},\mathbf{v}\in\mathbb{R}^d, \tag{1.1}$$

where the Hamiltonian H is given by

$$H(\mathbf{x},\mathbf{v}) = c(\mathbf{x})|\mathbf{v}| \tag{1.2}$$

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with $c(\mathbf{x})$ being the local wave speed of the medium. The bicharacteristics of equation (1.1) satisfy the Hamiltonian system:

$$\frac{d\mathbf{x}}{dt} = c\frac{\mathbf{v}}{|\mathbf{v}|}, \quad \frac{d\mathbf{v}}{dt} = -\nabla_{\mathbf{x}} c|\mathbf{v}|. \tag{1.3}$$

The Liouville equation (1.1) arises in the phase space description of geometrical optics. It can be derived as the high frequency limit of the wave equation

$$u_{tt} - c(\mathbf{x})^2 \Delta u = 0 \tag{1.4}$$

via the Wigner transform [1–3]. It is also the basis of computing multi-valued physical observables [4–8].

We are particularly interested in the case when $c(\mathbf{x})$ contains discontinuities due to different refractive indices at different media. The discontinuity corresponds to an interface, at which incoming waves can be partially transmitted and reflected. Against this background, much work has been done in the past both analytically [9–12] and numerically [13–18]. Numerically this problem consists of three challenges:

(1) One needs to provide an interface condition at the discontinuities of $c(\mathbf{x})$ to account for partial transmissions and reflections. This was first done in [13, 14], where the interface conditions (consistent to Snell's law) were built into numerical fluxes — the so-called Hamiltonian-preserving (HP) scheme.

(2) Due to the transmissions and reflections, f becomes discontinuous which then propagates linearly along the bicharacteristics (1.3). These are linear (contact) discontinuities that will be smeared by a typical finite difference or finite volume method, which necessarily contains numerical viscosities to suppress numerical oscillations across the discontinuities, with the smearing zone increases with time [19].

(3) The Liouville equation arising in geometrical optics or semiclassical limit involves measure-valued initial condition of delta-function shape:

$$f(\mathbf{x}, \mathbf{v}, 0) = \rho_0(\mathbf{x})\delta(\mathbf{v} - \mathbf{u}_0(\mathbf{x})). \tag{1.5}$$

The solution at later time remains measure-valued (with finite or even infinite number of concentrations corresponding to caustics in the physical space). For this type of problem, finite difference/volume methods usually produce poor quality results as the approximate delta functions are quickly smeared out due to numerical dissipation. The level set method proposed in [20] decomposes f into ϕ and ψ_i $(i = 1, \dots, d)$, where ϕ and ψ_i solve the same Liouville equation with initial data

$$\phi(\mathbf{x}, \mathbf{v}, 0) = \rho_0(\mathbf{x}), \quad \psi_i(\mathbf{x}, \mathbf{v}, 0) = v_i - u_{0i}(\mathbf{x}) \tag{1.6}$$

respectively. The density and averaged slowness can then be recovered by taking the moments of f:

$$\rho(\mathbf{x}, t) = \int_{\mathbb{R}^d} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} = \int_{\mathbb{R}^d} \phi\, \Pi_{i=1}^d \delta(\psi_i)\, d\mathbf{v}, \tag{1.7}$$

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{\rho(\mathbf{x}, t)} \int_{\mathbb{R}^d} f(\mathbf{x}, \mathbf{v}, t)\mathbf{v}\, d\mathbf{v} = \frac{1}{\rho(\mathbf{x}, t)} \int_{\mathbb{R}^d} \phi\, \Pi_{i=1}^d \delta(\psi_i)\mathbf{v}\, d\mathbf{v}. \tag{1.8}$$

This approach allows the computation of bounded rather than measure-valued solutions, which greatly enhances the numerical resolution. However, as pointed out in [8, 14], it only readily