Journal of Computational Mathematics
Vol.31, No.6, 2013, 549–572.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1307-m3955

# HIGHLY OSCILLATORY DIFFUSION-TYPE EQUATIONS*

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**Abstract**

We explore new asymptotic-numeric solvers for partial differential equations with highly oscillatory forcing terms. Such methods represent the solution as an asymptotic series, whose terms can be evaluated by solving non-oscillatory problems and they guarantee high accuracy at a low computational cost. We consider two forms of oscillatory forcing terms, namely when the oscillation is in time or in space: each lends itself to different treatment. Numerical examples highlight the salient features of the new approach.

*Mathematics subject classification:* 65M70, 35B05, 65M06, 35C20, 42A16.
*Key words:* Diffusion-type PDEs, High oscillation, Asymptotic expansions, Modulated Fourier expansions.

## 1. Introduction

Partial differential equations with highly oscillatory forcing terms arise in various branches of science and engineering. In particular, in modern communication systems, high frequencies and signals of widely-varying frequency content abound and present a serious challenge to existing numerical solvers [28, 29]. This is because the highly oscillatory behaviour of the problem compels the use of an exceedingly small step size with such methods. This results in significant accumulation of error and a prohibitive computational workload. The purpose of this paper is to address this issue and develop a numerical methodology which allows for an exceedingly accurate, yet affordable, discretization of partial differential equations with highly oscillatory forcing terms.

The numerical approach presented in this paper is based on the combination of asymptotic and numerical techniques. It involves asymptotic expansions in inverse powers of the oscillatory parameter, $\omega$ and numerical discretization of non-oscillatory partial differential equations which

* Received November 25, 2011 / Revised version received April 23, 2013 / Accepted July 9, 2013 / Published online October 18, 2013 /

are obtained in the course of the asymptotic expansion. In some situations, this numerical effort can be further reduced by using analytic results.

The core technique has been employed for the numerical approximation of ordinary differential equations that contain highly-oscillatory forcing terms in [12–15]. For example, the solution of the ordinary differential system

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) + \sum_{n=-\infty}^{\infty} \mathbf{b}_n(t)\mathrm{e}^{\mathrm{i}n\omega t}, \quad t \ge 0, \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{1.1}$$

can be expanded asymptotically in the form

$$\mathbf{y}(t) = \mathbf{p}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} \mathbf{p}_{r,n}(t)\mathrm{e}^{\mathrm{i}n\omega t}, \quad t \ge 0, \tag{1.2}$$

where the functions $\mathbf{p}_{r,n}$ are independent of $\omega$, hence non-oscillatory, and can be obtained for $n = 0$ by solving a non-oscillatory ordinary differential equation and by recursion for $n \neq 0$ [15].

The asymptotic expansion (1.2) has been shown to have very important benefits compared to standard discretization methods for ordinary differential equations. Firstly, the method is considerably more efficient for large values of the oscillatory parameter. Secondly, the computational effort is independent of the value of that parameter [12]. This success motivates our effort to extend it from systems of the form (1.1) to partial differential equations, the subject matter of this paper.

The current paper concentrates on the diffusion equation with Dirichlet or Neumann boundary conditions in the interval $[-1, 1]$ and it is concerned with two types of oscillatory terms, namely when the oscillation occurs in time or in space. We demonstrate that a model similar to (1.2) falls short of describing a solution of a diffusion equation with a highly-oscillatory forcing term but a considerably more complicated *ansatz* is equal to this task. We also observe an interesting phenomenon: while in the case of time-like oscillations in the forcing term, the solution, similar to (1.2), consists of a non-oscillatory component overlaid with small amplitude oscillations, the solution for a forcing term oscillating in space is itself non-oscillatory!

The extension of the methodology underlying (1.2), blending asymptotic and numerical techniques, into the realm of partial differential equations is far from simple. In particular, it necessitates the solution of initial-boundary value problems which, although non-oscillatory, present us with a major challenge once we wish to obtain high accuracy. This is further elaborated in the sequel.

Interesting applications of partial differential equations with highly oscillatory forcing terms are not restricted to the diffusion equation: a case in point is computational electronics, where the differential operator is hyperbolic [29]. We can also remark that Modified Fourier Expansions have been developed to treat analytical and numerical problems in partial differential equations [6–11]. However, this type of expansion uses a different kind of approach and it has been used in a different kind of problem setting to the one we address in this paper. In the current paper we consider a special case, while endeavouring to establish a general framework relevant to other highly oscillatory equations.

The theory of Laplace–Dirichlet and Laplace–Neumann expansions in parallelepipeds is quite comprehensively understood (e.g., [1]). We recognise that the extension of the expansions in this paper to several dimensions is likely to result in fairly unpleasant expressions, but this is a technical, rather than conceptual difficulty. We note that such expansions have been also