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A RELAXED HSS PRECONDITIONER FOR SADDLE POINT PROBLEMS FROM MESHFREE DISCRETIZATION*

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Abstract

In this paper, a relaxed Hermitian and skew-Hermitian splitting (RHSS) preconditioner is proposed for saddle point problems from the element-free Galerkin (EFG) discretization method. The EFG method is one of the most widely used meshfree methods for solving partial differential equations. The RHSS preconditioner is constructed much closer to the coefficient matrix than the well-known HSS preconditioner, resulting in a RHSS fixed-point iteration. Convergence of the RHSS iteration is analyzed and an optimal parameter, which minimizes the spectral radius of the iteration matrix is described. Using the RHSS preconditioner to accelerate the convergence of some Krylov subspace methods (like GMRES) is also studied. Theoretical analyses show that the eigenvalues of the RHSS preconditioned matrix are real and located in a positive interval. Eigenvector distribution and an upper bound of the degree of the minimal polynomial of the preconditioned matrix are obtained. A practical parameter is suggested in implementing the RHSS preconditioner. Finally, some numerical experiments are illustrated to show the effectiveness of the new preconditioner.

Mathematics subject classification: 65F10.
Key words: Meshfree method, Element-free Galerkin method, Saddle point problems, Preconditioning, HSS preconditioner, Krylov subspace method.

1. Introduction

In recent years, meshfree (or meshless) methods have been developed rapidly as a class of potential computational techniques for solving partial differential equations. In the meshfree method, it does not require a mesh to discretize the problem domain, and the approximate solution is constructed entirely on a set of scattered nodes. A lot of meshfree methods have been proposed; see [27] for a general discussion. In this paper, we mainly consider the element-free Galerkin method [11], which is one of the most widely used meshfree methods.

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The EFG method is almost identical to the conventional finite element method (FEM), as both of them are based on the Galerkin formulation, and employ local interpolation/approximation to approximate the trial function. The EFG method requires only a set of distributed nodes on the problem domain, while elements are used in the finite element method. Other key differences between the EFG method and the FEM method lie in the interpolation methods, integration schemes and in the enforcement of essential boundary conditions. The EFG method employs the moving least squares (MLS) approximation method to approximate the trial functions. One disadvantage of MLS approximation is that the shape functions obtained are lack of Kronecker delta function property, unless the weight functions used in the MLS approximation are singular at nodal points. Therefore, the essential boundary conditions in the EFG method can not be easily and directly enforced. Several approaches have been proposed for imposing the essential boundary conditions in the EFG method, such as Lagrange multiplier (LM) method [11, 28], penalty method [33], augmented Lagrangian (AL) method [31], coupled method [22] and so on. Using independent Lagrange multipliers to enforce essential boundary conditions is common in structural analysis when boundary conditions can not be directly applied. However, this method leads to a linear system of saddle point type and increases the number of unknowns. The penalty method is very simple to be implemented and yields a symmetric positive definite stiffness matrix, but the penalty parameter must be chosen appropriately. Moreover, the accuracy of the penalty method is less than that of the Lagrange multiplier method in general. The augmented Lagrangian method uses a generalized total potential energy function to impose essential boundary conditions. The augmented Lagrangian regularization used in the AL method is composed of the sum of the pure Lagrangian term and the penalty term. In fact, the AL method combines the LM method and the penalty method, but it leads to a better matrix structure than the LM method; see discussion in Section 2 and [31]. For other methods studied for imposing essential boundary conditions in the EFG method; see [22, 27] and references therein.

In this paper, we study the iterative solutions of large and sparse linear systems of equations arising from the EFG method with the AL method imposing essential boundary conditions. The discrete linear system of equations has the following block 2×2 form

$$\mathcal{A}x \equiv \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{1.1}$$

where $A = K + G \in \mathbb{R}^{n\times n}$ (K is the stiffness matrix and G is obtained from penalty term), $B \in \mathbb{R}^{m\times n}$ is obtained from Lagrangian term, $u \in \mathbb{R}^n$ is the approximation solution at nodes, $\lambda \in \mathbb{R}^m$ is the Lagrange multiplier, $f = f_1 + f_2 \in \mathbb{R}^n$ (f_1 is the stiffness vector and f_2 is obtained from penalty term), $g \in \mathbb{R}^m$ and $m \le n$. n and m are related to the number of nodes in global problem domain and the number of nodes in essential boundary, respectively. In general, A is symmetric and positive definite and B has full rank. Under those conditions, we know that the solution of (1.1) exists and is unique.

The linear system (1.1) can be regarded as the saddle point problem. It frequently arises from computational fluid dynamics, mixed finite element of elliptic PDEs, constrained optimization, constrained least-squares problem and so on. In recent years, saddle point problems have been received considerable attention. A large amount of work has been devoted to developing efficient algorithms for solving saddle point problems. When the matrix blocks A and B are large and sparse, iterative methods become more attractive than direct methods for solving the saddle point problems (1.1), but direct methods play an important role in the form of