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A SPARSE-GRID METHOD FOR MULTI-DIMENSIONAL BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

A sparse-grid method for solving multi-dimensional backward stochastic differential equations (BSDEs) based on a multi-step time discretization scheme [31] is presented. In the multi-dimensional spatial domain, i.e. the Brownian space, the conditional mathematical expectations derived from the original equation are approximated using sparse-grid Gauss-Hermite quadrature rule and (adaptive) hierarchical sparse-grid interpolation. Error estimates are proved for the proposed fully-discrete scheme for multi-dimensional BSDEs with certain types of simplified generator functions. Finally, several numerical examples are provided to illustrate the accuracy and efficiency of our scheme.

Mathematics subject classification: 60H10, 60H35, 65C10, 65C20, 65C50.
Key words: Backward stochastic differential equations, Multi-step scheme, Gauss-Hermite quadrature rule, Adaptive hierarchical basis, Sparse grids.

1. Introduction

We consider the following backward stochastic differential equation (BSDE)

$$\begin{cases} -dy_t = f(t, y_t, z_t)dt - z_t dW_t, \quad t \in [0, T), \\ y_T = \xi, \end{cases} \tag{1.1}$$

where T is a fixed positive number, W_t is the standard d-dimensional Brownian motion defined on a complete, filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0\leq t\leq T})$, $f(t, y_t, z_t)$ is an adapted stochastic process with respect to $\{\mathcal{F}_t\}$ $(0 \leq t \leq T)$ for each (y_t, z_t), and ξ is an $\{\mathcal{F}_T\}$ measurable random variable. The existence and uniqueness of the solution of the BSDE (1.1) were proved by Pardoux and Peng in [20]. Since then, BSDEs and their solutions have been extensively studied. In [22], Peng obtained a direct relation between forward-backward stochastic differential equations and partial differential equations and then, in [21], he also derived a maximum principle for stochastic control problems. Many important properties of BSDEs and their applications in finance were studied by Karoui et al. in [8].

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Because analytical solutions of BSDEs are often very difficult to obtain, approximate numerical solutions of BSDEs become highly desired in practical applications. There are mainly two types of numerical methods for BSDEs. One is based on the relation between the forward-backward stochastic differential equations (FBSDEs) and corresponding parabolic partial differential equation (PDEs) [13, 14, 22]; the other is directly based on BSDEs or FBSDEs [2, 3, 6, 7, 10, 12, 23, 27, 28, 30, 31]. Zhao et al. proposed a θ-scheme for BSDEs in [28]; in [29], it was extended to a generalized θ-scheme. In [31], a stable multi-step scheme was proposed which is a highly accurate numerical method for BSDEs. Note that for the second type of numerical methods, approximating spatial derivatives at different time-space points for the case of PDEs is converted to approximating conditional mathematical expectations with Gaussian kernels centered at different time-space points.

It should be noted that the BSDEs used in practice usually involve a multi-dimensional Brownian motion, such as the option pricing problem with multiple underlying assets. Existing numerical methods for BSDEs can be theoretically extended to the multi-dimensional cases; however, the computational cost may be unaffordable due to the so-called *curse of dimensionality*. The most popular approach to solving multi-dimensional BSDEs is the Monte Carlo method [3, 28] that is very easy to implement. However, the convergence rate is typically very slow, although having a mild dependence on the dimensionality. Thus, an accurate and efficient numerical method for solving multi-dimensional BSDEs is highly desired in the BSDE community.

In this paper, we extend the multi-step method in [31] using the sparse-grid method for solving multi-dimensional BSDEs. As discussed in [31], the target BSDE (1.1) is discretized by the multi-step scheme in the time direction. In the spatial domain, a quadrature rule is needed to approximate all the conditional mathematical expectations (multi-dimensional integrals) and an interpolation scheme is also needed to evaluate the integrands of the expectations at non-grid quadrature points. The sparse-grid method is highly suitable for the multi-step scheme because it has been demonstrated to be effective and efficient in dealing with multi-dimensional interpolation and quadrature [1, 4, 9, 11, 16–19]. Sparse-grid interpolantion (or quadrature rule) resulting from the Smolyak algorithm depends weakly on dimensionality so the computational expense can be significantly reduced; however, the accuracy can be preserved up to a logarithmic factor compared with tensor-product interpolantion (or quadrature rule). On the other hand, the multi-step method is also highly suitable for the sparse-grid method because no spatial derivatives are involved in the multi-step scheme and the solution can be obtained without solving a linear system. In comparison, the sparse-grid method can be potentially used together with finite difference or finite element method to solve the associated parabolic PDE instead solving the BSDE directly; in this case, spatial derivatives need to be discretized on sparse grids such that the resulting linear system may have stability or conditioning issues and a CFL condition needs to be satisfied for solving the time-dependent problem. In [24, 25], a spectral sparse-grid method was proposed for elliptic problems, which does not have severe stability or conditioning issues. However, in this paper, those issues on linear systems are completely avoided in our method and the CFL condition is not needed either. In addition, the sparse-grid method is also suitable for the θ-scheme in [28] and the generalized θ-scheme in [29]; we focus on the multi-step method because it is more accurate than the other two schemes in the time direction.

The main contributions in this paper are as follows:

- propose a fully-discrete scheme with the sparse-grid method for multi-dimensional BSDEs;