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OPTIMALITY OF LOCAL MULTILEVEL METHODS FOR ADAPTIVE NONCONFORMING P1 FINITE ELEMENT METHODS*

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Abstract

In this paper, a local multilevel product algorithm and its additive version are considered for linear systems arising from adaptive nonconforming P1 finite element approximations of second order elliptic boundary value problems. The abstract Schwarz theory is applied to analyze the multilevel methods with Jacobi or Gauss-Seidel smoothers performed on local nodes on coarse meshes and global nodes on the finest mesh. It is shown that the local multilevel methods are optimal, i.e., the convergence rate of the multilevel methods is independent of the mesh sizes and mesh levels. Numerical experiments are given to confirm the theoretical results.

Mathematics subject classification: 65F10, 65N30.
Key words: Local multilevel methods, Adaptive nonconforming P1 finite element methods, Convergence analysis, Optimality.

1. Introduction

Multigrid methods and other multilevel preconditioning methods for nonconforming finite elements have been studied by many researchers (cf. [4–7,16,22–25,27,30–32,36,38]). The BPX framework developed in [4] provides a unified convergence analysis for nonnested multigrid methods. Duan *et al.* [16] extended the result to general V-cycle nonnested multigrid methods, but only the case of full elliptic regularity was considered. Besides, Brenner [7] established a framework for the nonconforming V-cycle multigrid method under less restrictive regularity assumptions. All the above convergence results for nonconforming multigrid methods are based on the requirement of a sufficiently large number of smoothing steps at each level. For multilevel preconditioning methods, Oswald developed a hierarchical basis multilevel method [23] and a BPX-type multilevel preconditioner [24] for nonconforming finite elements. On the other

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hand, Vassilevski and Wang [30] presented multilevel algorithms with only one smoothing step per level. These multilevel algorithms may be considered as successive subspace correction methods (SSC) (cf. [34] for details). They are completely different from standard nonconforming multigrid methods [5]. By using the well-known Schwarz framework, a uniform convergence result has been obtained. The idea that the conforming finite element spaces are contained in their nonconforming counterparts is essential in the analysis of the multilevel algorithms (cf. [30] for details). In this paper, we will use this idea to design optimal multilevel methods for adaptive nonconforming P1 element methods (ANFEM). We note that Hoppe and Wohlmuth [18] considered multilevel preconditioned conjugate gradient methods for nonconforming P1 finite element approximations with respect to adaptively generated hierarchies of nonuniform meshes based on residual type a posteriori error estimators.

Recent studies (cf., e.g., [2, 11–13, 19, 21, 28]) indicate optimal convergence properties of adaptive conforming and nonconforming finite element methods. Therefore, in order to achieve an optimal numerical solution, it is imperative to study efficient iterative algorithms for the solution of linear systems arising from adaptive finite element methods (AFEM). Since the number of degrees of freedom N per level may not grow exponentially with mesh levels, as Mitchell has pointed out in [20] for adaptive conforming finite element methods, the number of operations used for multigrid methods with smoothers performed on all nodes can be as bad as $O(N^2)$, and a similar situation may also occur in the nonconforming case.

For adaptive conforming finite element methods, the optimality of local multilevel methods for 2D and 3D $H^1(\Omega)$-elliptic problems has been studied in [17, 33, 35, 37]. The hierarchy of meshes used in the local multilevel methods can be obtained either by successive adaptive refinement of an initial coarse mesh or by successive coarsening of a fine mesh. Wu and Chen [33] applied the adaptively refined hierarchy of meshes generated by the newest vertex bisection and obtained uniform convergence for the multigrid V-cycle algorithm with local Gauss-Seidel smoother in 2D. The optimal multigrid methods developed by Xu, Chen and Nochetto [35] are based on the reconstruction of hierarchy of meshes. There are some assumptions of this strategy on the initial mesh and the fine mesh to guarantee that the compatible patches of meshes do exist (cf. [35]). We do not reconstruct a virtual refinement hierarchy of meshes in our algorithms, but use the hierarchy generated by the ANFEM. We also note that Dahmen and Kunoth [15] proved the optimality of BPX preconditioner for piecewise linear finite elements on the quasi-uniform meshes and the nonuniform meshes generated by red-green refinement. Brenner, Cui and Sung [8] proved the uniform convergence of W-cycle multigrid algorithm with sufficiently large number of smoothing steps for the symmetric interior penalty method on graded meshes in 2D. To our knowledge, so far there does not exist an optimal multilevel method for nonconforming finite element methods on locally refined meshes. Indeed, there are two difficulties in the theoretical analysis which need to be overcome. First, since the multilevel spaces are nonnested in this situation, we should consider how to design a stable decomposition of the finest nonconforming finite element space. The second difficulty is how to establish the strengthened Cauchy-Schwarz inequality on nonnested multilevel spaces. In this paper, we will construct a special prolongation operator from the coarse space to the finest space, and obtain the key global strengthened Cauchy-Schwarz inequality. Two multilevel methods, the product and additive version, are proposed. Applying the well-known Schwarz theory (cf. [29]), we show that local multilevel methods for adaptive nonconforming P1 finite element methods are optimal.

The remainder of this paper is organized as follows: In Section 2, we introduce some nota-