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A STABILIZED EQUAL-ORDER FINITE VOLUME METHOD FOR THE STOKES EQUATIONS *

Wanfu Tian
School of Science, Shenyang Aerospace University, Shenyang 110136, China
Email: tianwf@163.com
Liqiu Song
Institute of Mathematics, Jilin University, Changchun 130012, China
Email: gslautumn@yahoo.com.cn
Yonghai Li
School of Mathematics, Jilin University, Changchun 130012, China
Email: yonghai@jlu.edu.cn

Abstract

We construct a new stabilized finite volume method on rectangular grids for the Stokes equations. The lowest equal-order conforming finite element pair (piecewise bilinear velocities and pressures) and piecewise constant test spaces for both the velocity and pressure are employed in this method. We show the stability of this method and prove first optimal rate of convergence for the velocity in the H^1 norm and the pressure in the L^2 norm. In addition, a second order optimal error estimate for the velocity in the L^2 norm is derived. Numerical experiments illustrating the theoretical results are included.

Mathematics subject classification: 65N08, 65N15, 65N30, 76D05.
Key words: Stokes equations, Equal-order finite element pair, Finite volume method, Error estimate.

1. Introduction

Finite volume method (FVM) [2,8,9,13,26,27,32], also called generalized difference method, covolume method, or box scheme, has been widely used in computational fluid dynamics and practical fluid mechanics. In general, the programming effort in implementing the finite volume method is usually simpler than the finite element method (FEM). The finite volume method discretizations provide reasonable approximations for the Stokes problems. Many papers were devoted to develop the finite volume method and establish its error analysis for the Stokes equations, for example, see [10–12, 23, 28, 29, 33].

The lowest equal-order finite element pair for the Stokes equations have already attracted much attention [1,3,5,7,16,18,20–24,30] because of their simplicity and attractive computational properties. Since the equal-order finite element pairs hold an identical degree distribution for both the velocity and pressure, they are computationally efficient in multigrids and parallel processing. However, it is well known that the equal-order finite element pairs do not satisfy the inf-sup condition. In order to counteract the lack of inf-sup stability, one possible remedy is to modify the variational formulation associated with the Stokes equations by adding a stabilization term.

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Recently, Li and Chen [23] have developed and analyzed a stabilized finite volume method for the Stokes equations on triangular grids. The lowest equal-order conforming finite element pair (piecewise linear velocities and pressures) are employed in their method. By the relationship between their method and a stabilized finite element method they derived the optimal error estimates for both the velocity and pressure.

In this paper, we study a new stabilized finite volume method for the Stokes equations on rectangular grids with the lowest equal-order conforming finite element pair(piecewise bilinear velocities and pressures). We consider the following stationary Stokes problem in an axiparallel domain $\Omega \subset \mathbb{R}^2$

$$-\lambda\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\text{div}\mathbf{u} = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\mathbf{u} = (u^1, u^2)$ stands for fluid velocity, p the pressure, $\mathbf{f}$ is a given external force, and $\lambda > 0$ denotes the viscosity of the fluid. Set

$$\mathbf{V} = H_0^1(\Omega)^2, \quad W = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q d\Omega = 0 \right\}. \tag{1.4}$$

Define

$$A(\mathbf{u}, \mathbf{v}) = \lambda \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} d\Omega, \tag{1.5}$$

$$C(\mathbf{v}, p) = \int_\Omega \mathbf{v} \cdot \nabla p d\Omega, \quad B(\mathbf{u}, q) = -\int_\Omega q \text{div } \mathbf{u} d\Omega. \tag{1.6}$$

It is well known that $C(\mathbf{v}, q) = B(\mathbf{v}, q)$, then the associated variational formulation of (1.1)-(1.3) is to seek a pair $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{1.7}$$

$$B(\mathbf{u}, q) = 0, \quad \forall q \in W. \tag{1.8}$$

The above weak formulation (1.1)-(1.3) can be also written as follows:

$$\begin{aligned} L(\mathbf{u}, p; \mathbf{v}, q) &:= A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) + B(\mathbf{u}, q) \\ &= (\mathbf{f}, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times W. \end{aligned} \tag{1.9}$$

In the case of rectangular partition, since the bilinear form C for the finite volume method is no longer equal to the bilinear form B for the finite element method when the lowest equal-order finite element pair are employed, the finite volume method for the Stokes problem (1.1)-(1.3) can not be written as the standard form. To compensate for this deficiency, we discretize Eq. (1.2) using the finite volume method instead of the finite element method, which is different from the classical mixed methods [4,6,11,12,17,19,29]. Moreover, in order to stabilize this system, with the idea of [3,24] we introduce a stabilization term using a local polynomial pressure projection on dual elements. We will show that our method is unconditionally stable, and achieve optimal accuracy. Moreover, numerical experiments confirm the theoretical results.

The remainder of this paper is organized as follows. In the next section we introduce some notations which will be used throughout the paper and recall the stabilized finite element