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INVERSE BORN SERIES FOR SCALAR WAVES*

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Abstract

We consider the inverse scattering problem for scalar waves. We analyze the convergence of the inverse Born series and study its use in numerical simulations for the case of a spherically-symmetric medium in two and three dimensions.

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1. Introduction

The inverse scattering problem (ISP) for scalar waves consists of recovering the spatially-varying index of refraction (or scattering potential) of a medium from measurements of the scattered field. This problem is of fundamental interest and considerable applied importance. There is a substantial body of work on the ISP that has been been comprehensively reviewed in [7–9]. In particular, much is known about theoretical aspects of the problem, especially concerning the issues of uniqueness and stability. There has also been significant effort devoted to the development of techniques for image reconstruction, including optimization, qualitative and direct methods. There is also closely related work in which small-volume expansions have been used to reconstruct the scattering properties of small inhomogeneities. The corresponding reconstruction algorithms have been implemented and their stability analyzed as a function of the signal-to-noise-ratio of the data [1–6].

In previous work, we have proposed a direct method to solve the inverse problem of optical tomography that is based on inversion of the Born series [11–13]. In this approach, the solution to the inverse problem is expressed as an explicitly computable functional of the scattering data. In combination with a spectral method for solving the linear inverse problem, the inverse Born series leads to a fast image reconstruction algorithm with analyzable convergence, stability and error.

In this paper we apply the inverse Born series to the ISP for scalar waves. We characterize the convergence, stability and approximation error of the method. We also illustrate its use in numerical simulations. We find that the series converges rapidly for low contrast objects. As the contrast is increased, the higher order terms systematically improve the reconstructions until, at sufficiently large contrast, the series diverges.

The remainder of this paper is organized as follows. In Section 2, we construct the Born series for scalar waves. We then derive various estimates that are later used to study the

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convergence of the inverse Born series. The inversion of the Born series is taken up in Section 3. In Section 4, the forward operators in the Born series are calculated for the case of radially varying media. Exact solutions to the problem of scattering by spheres and annuli are discussed in Section 5. These results are used as forward scattering data for numerical reconstructions, which are shown in Section 6. Finally, our conclusions are presented in Section 7.

2. Born Series

We consider the propagation of scalar waves in $\mathbb{R}^n$ for $n \geq 2$. The field u obeys the equation

$$\nabla^2 u(x) + k^2(1+\eta(x))u(x) = 0. \tag{2.1}$$

It will prove useful to decompose the field into the sum of an incident field and a scattered field:

$$u = u_i + u_s. \tag{2.2}$$

The incident field will be taken to be a plane wave of the form

$$u_i(x) = e^{ikx\cdot\xi} \ , \tag{2.3}$$

where k is the wave number and $\xi \in S^{n-1}$ is the direction in which the incident wave propagates. The scattered field u_s satisfies

$$\nabla^2 u_s(x) + k^2 u_s(x) = -k^2\eta(x)u(x) \tag{2.4}$$

and obeys the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - iku_s\right) = 0 \ . \tag{2.5}$$

The function $\eta(x)$ is the perturbation of the squared refractive index, which is assumed to be supported in a closed ball B_a of radius a. The solution u can be expressed as the solution to the Lippmann-Schwinger integral equation

$$u(x) = u_i(x) + k^2\int_{B_a} G(x,y)u(y)\eta(y)dy \ , \tag{2.6}$$

where the Green's function G satisfies the equation

$$\nabla_x^2 G(x,y) + k^2 G(x,y) = -\delta(x-y) \ . \tag{2.7}$$

Applying a fixed point iteration to (2.6), beginning with the incident wave, gives the well known Born series for the total field u

$$\begin{aligned} u(x) =& u_i(x) + k^2\int_{B_a} G(x,y)\eta(y)u_i(y)dy \\ &+ k^4\int_{B_a\times B_a} G(x,y)\eta(y)G(y,y^{'})\eta(y^{'})u_i(y^{'})dydy^{'} + \cdots \ . \end{aligned} \tag{2.8}$$

Let us define the scattering data

$$\phi = u_i - u. \tag{2.9}$$