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A DISCONTINUOUS GALERKIN METHOD FOR THE FOURTH-ORDER CURL PROBLEM*

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Abstract

In this paper, we present a discontinuous Galerkin (DG) method based on the Nédélec finite element space for solving a fourth-order curl equation arising from a magnetohydrodynamics model on a 3-dimensional bounded Lipschitz polyhedron. We show that the method has an optimal error estimate for a model problem involving a fourth-order curl operator. Furthermore, some numerical results in 2 dimensions are presented to verify the theoretical results.

Mathematics subject classification: 65N30.
Key words: Fourth-order curl problem, DG method, Nédélec finite element space, Error estimate.

1. Introduction

Magnetohydrodynamics (MHD) equations describe the macroscopic dynamics of electrical fluid that moves in a magnetic field. The MHD model is governed by Navier-Stokes equations coupled with Maxwell equations through Ohm's law and the Lorentz force. As an example, a resistive MHD system is described by the following equations:

$$\begin{cases} \rho(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla p = \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \mu\triangle\boldsymbol{u}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{B}_t - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) \\ \qquad = -\frac{\eta}{\mu_0}\nabla\times(\nabla\times\boldsymbol{B}) - \frac{d_i}{\mu_0}\nabla\times((\nabla\times\boldsymbol{B})\times\boldsymbol{B}) - \frac{\eta_2}{\mu_0}(\nabla\times)^4\boldsymbol{B}, \\ \nabla\cdot\boldsymbol{B} = 0, \end{cases} \tag{1.1}$$

where ρ is the mass density, $\boldsymbol{u}$ is the velocity, p is the pressure, $\boldsymbol{B}$ is the magnetic induction field, η is the resistivity, η_2 is the hyper-resistivity, μ_0 is the magnetic permeability of free space,

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and μ is the viscosity. The primary variables in MHD equations are the fluid velocity $\boldsymbol{u}$ and the magnetic field $\boldsymbol{B}$.

MHD models have wide applications in thermonuclear fusion, plasma physics, geophysics, and astrophysics. The mathematical modeling and numerical simulation of MHD have been the subject of considerable research effort in the past few decades, such that various numerical algorithms have been proposed in MHD simulations. In order to solve MHD equations that contain a fourth-order term, we will focus on using a DG method.

DG methods are effective methods in solving partial differential equations. As far back as 1973, Reed and Hill [33] proposed a DG method for the hyperbolic equation. Since then, DG methods have been used widely to solve hyperbolic problems (see, e.g., [14–17]) and elliptic problems (see, e.g., [1, 2, 10–12, 19, 35]). At the same time, they have important applications to other problems such as Navier-Stokes equations (see,e.g., [8, 18]), Euler equation [38], and fractional diffusion problem [34]. The use of DG methods for elliptic problems can be traced back to the penalty method [28] and the interior penalty (IP) method [6]. A lot of work (see, e.g., [2, 4–7, 28, 35]) has been done on DG methods for second-order elliptic equations. For fourth-order elliptic problems, Baker [7] used an IP method to solve the biharmonic problem on 2-dimensional smooth domains. Engel et al. [20] combined concepts from the continuous Galerkin method, the DG method and stabilization techniques to approximate fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. In addition, Brenner and Sung [9] used an IP method to solve the biharmonic problem on 2-dimensional bounded polygonal domains. Xu and Shu [37] reviewed the works on local DG methods for high-order time-dependent problems.

Recently, DG methods have also been applied in the numerical simulation of Maxwell equations. In 2002, Perugia et al. [32] used an IP method for the time harmonic Maxwell equation. In 2004, Cockburn et al. [13] used a local divergence-free DG method for the Maxwell equation. In their study, the approximate solution is preserved divergence-free on each element, and computational costs are much lower than for standard DG methods. Moreover, Houston et al. [22] used a mixed DG method to approximate the Maxwell operator; Lu et al. [29] gave a DG method for the Maxwell equation with Debye-type dissipative material and artificial PML (perfectly matched layer) boundary. In 2005, Houston et al. applied an IP method [23] and a mixed DG method [24] for the indefinite time harmonic Maxwell equation. Recently, Li [26, 27] considered an interior penalty DG method for the time-dependent Maxwell equations in cold plasma.

In the numerical simulation of the MHD equations (1.1), it is necessary to design an efficient numerical discretization for a fourth-order curl problem. As is well-known, constructing a curl-curl-conforming element for the fourth-order curl problem is very difficult. Zheng, Hu, and Xu [39] used a nonconforming finite element method to solve fourth-order curl equations. Motivated by the IP method for the fourth-order elliptic problem [9], in this paper, we design a DG method for solving the fourth-order curl problem. The main feature of this scheme is that we can use the standard higher-order Nédélec finite element space.

In this paper, we begin by introducing the fourth-order curl model equation. According to the model problem, we establish the corresponding variational problem by introducing suitable function spaces. By showing that the trial function space is a Hilbert space, we give the well-posedness of the variational problem and a regularity result of the weak solution. Second, based on the standard higher-order Nédélec finite element space, we design a DG method for the fourth-order curl problem and prove the boundedness and coercivity of the discrete variational