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EXPLICIT ERROR ESTIMATES FOR COURANT, CROUZEIX-RAVIART AND RAVIART-THOMAS FINITE ELEMENT METHODS*

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Abstract

The elementary analysis of this paper presents explicit expressions of the constants in the a priori error estimates for the lowest-order Courant, Crouzeix-Raviart nonconforming and Raviart-Thomas mixed finite element methods in the Poisson model problem. The three constants and their dependences on some maximal angle in the triangulation are indeed all comparable and allow accurate a priori error control.

Mathematics subject classification: 65N15, 65N12, 65N30.
Key words: Error estimates, Conforming, Nonconforming, Mixed, Finite element method.

1. Introduction

Quantitative a priori error control for the three most popular lowest-order conforming, nonconforming, and mixed 2D finite element methods (FEMs) named after Courant, Crouzeix-Raviart, and Raviart-Thomas, depicted symbolically in Figure 1.1, is one of the most fundamental questions in the numerical analysis of partial differential equations (PDEs). For the Courant FEM and the Raviart-Thomas mixed FEM (MFEM), there exist elementwise interpolation operators I and I_F such that the error analysis consists in an estimate of the Lebesgue norms in the sense of

$$\|\nabla(v - Iv)\|_{L^2(T)} \le C(T) h_T \left\|D^2 v\right\|_{L^2(T)}$$

for some smooth function v with Hessian D^2v and the triangle T with diameter h_T. The point is that the constant $C(T)$ depends on the shape of the triangle but not on its size h_T. The textbook analysis is based on the Bramble-Hilbert lemma and so on some compact embeddings on a reference geometry [1, 2]. The transformation formula then leads to some estimate of $C(T)$ which is qualitative and can be quantified with the help of computer-justified values of some eigenvalue problem on the reference triangle, cf. e.g., [3] for a historic overview and the references quoted therein, in particular [4] for Courant and [5] for Raviart-Thomas FEM. This

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paper aims at direct elementary proofs of quantitative error estimates based on the Poincaré inequality with some known constant plus elementary integration by parts.

The situation is somewhat different for the nonconforming FEMs because the local interpolation error through the natural interpolation operator I_{NC} is very sharp, even optimal by some averaging property; but the global error is also driven by the interaction with the inconsistency. The standard textbook analysis employs some Strang-Fix type argument [1, 6] which leads to two contributions and gives the reader the impression that the error analysis is even more sensitive and perhaps even the scheme is more sensitive than the other two. In Braess [1] page 111 one can even find the hint that the Crouzeix-Raviart nonconforming FEM (NCFEM) is more sensitive with respect to large second order derivatives than the other two methods.

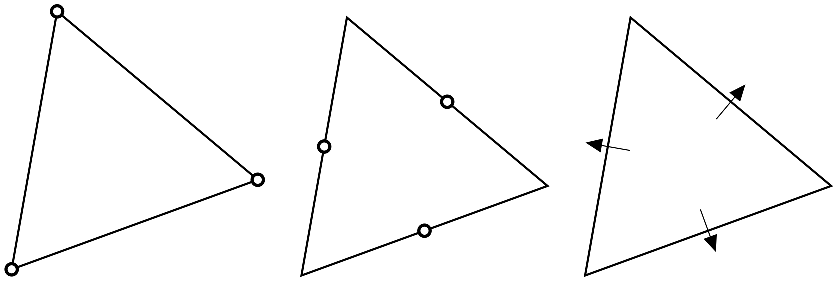

Fig. 1.1. Courant, Crouzeix-Raviart, and Raviart-Thomas FE.

This paper aims at a clarification by the comparison of the best known constants $C(T)$ for the three FEMs at hand. In fact, the constant

$$C(\alpha) := \sqrt{\frac{1/4 + 2/j_{1,1}^2}{1 - |\cos\alpha|}}, \tag{1.1}$$

for a maximal angle $0 < \alpha < \pi$ of a triangle T and the first positive root $j_{1,1}$ of the Bessel function J_1, and its maximum

$$C(\mathcal{T}) := \max_{T\in\mathcal{T}} C(\max\measuredangle T)$$

in a triangulation $\mathcal{T}$ of a 2D polygonal domain Ω play a dominant role. The main results of this paper are the explicit error estimates

$$|||u - u_C||| \le C(\mathcal{T}) \left\| h_{\mathcal{T}} D^2 u \right\|_{L^2(\Omega)}, \tag{1.2}$$

$$\|p - p_{RT}\|_{L^2(\Omega)} \le C(\mathcal{T}) \|h_{\mathcal{T}} Dp\|_{L^2(\Omega)}, \tag{1.3}$$

$$|||u - u_{CR}|||_{NC} \le \frac{1}{j_{1,1}} \operatorname{osc}(f,\mathcal{T}) + \sqrt{\frac{1}{j_{1,1}^2} + C(\mathcal{T})^2} \left\| h_{\mathcal{T}} D^2 u \right\|_{L^2(\Omega)} \tag{1.4}$$

for the Courant, Raviart-Thomas and Crouzeix-Raviart finite element approximations u_C, p_{RT} and u_{CR} in a simple Poisson model problem and the oscillations $\operatorname{osc}(f,\mathcal{T})$ defined in Section 6. In particular, the constants (which are upper bounds) have the same behaviour as the angles deteriorate with $\alpha \nearrow \pi$. The above estimate for the NCFEM displays the perturbation result for an arbitrary L^2 function f as a right-hand side in the Poisson model problem and thereby corrects and sharpens a corresponding error analysis in [7]. The technique here bypasses the