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A HIGHER-ORDER EULERIAN-LAGRANGIAN LOCALIZED ADJOINT METHOD FOR TWO-DIMENSIONAL UNSTEADY ADVECTION-DIFFUSION PROBLEMS*

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Abstract

We present a higher-order in-space characteristic method for the solution of the transient advection diffusion equations in two space dimensions. This method uses biquadratic trial and test functions within the framework of the Eulerian-Lagrangian localized Adjoint Methods (ELLAM). It therefore maintains the advantages of previous ELLAM schemes. Namely, it treats general boundary conditions naturally in a systematic manner, conserves mass, and symmetrizes the governing transport equations. Moreover, it generates accurate numerical solutions even if large time steps are used in the simulation. Numerical experiments are presented to illustrate the performance of this method and establish its order of convergence numerically.

Mathematics subject classification: 65D30.
Key words: Advection-diffusion equations, Characteristic methods, Eulerian-Lagrangian methods, Biquadratic interpolation.

1. Introduction

Advection-diffusion equations arise in many models in Science and Engineering and describe a wide range of natural phenomena characterized by moving fronts. In fluid dynamics, for example, the movement of a solute in groundwater is described by such an equation. These equations are known to present serious numerical difficulties especially when the magnitude of advection is relatively large. Standard finite difference and finite element methods fail to provide qualitatively correct solutions for this class of equations even with lower accuracy requirements. More specifically, these methods generate solutions which suffer from numerical artifacts including non-physical spurious oscillations and numerical diffusion that smears out sharp fronts of the solution where important chemistry and physics take place.

A number of specialized methods have been developed to resolve these difficulties. They generally can be classified as *Eulerian methods* or *characteristic methods*. Eulerian methods use fixed spatial grids and incorporate upstream weighting or some other dissipation techniques in their formulations [3, 7, 10, 11, 14, 15]. Thus, they can eliminate the non-physical oscillations present in the standard finite difference and finite element methods. Some of the Eulerian methods, such as the Godunov scheme, the total variation diminishing (TVD) schemes, and the ENO and WENO schemes, can resolve shock discontinuities from nonlinear hyperbolic conservation laws.

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Characteristic methods, on the other hand, make use of the hyperbolic nature of the governing equation by utilizing tracking along the characteristics to treat the advective part of the governing equation [8,12]. These methods symmetrize the governing equations and significantly reduce the temporal truncation errors. Thus, they allow large time steps to be used in the numerical simulations without any sacrifice in accuracy, and lead to a greatly improved efficiency. However, most characteristic methods have difficulties in treating general boundary conditions and fail to conserve mass. The Eulerian-Lagrangian localized adjoint method (ELLAM) was proposed by Celia *et. al.,* [6] as an alternative characteristic formulation that alleviates the difficulties mentioned. This ELLAM formalism provides a general characteristic solution procedure for advection-diffusion equations and a consistent framework for conserving mass and treating general boundary conditions.

The original ELLAM method is for the solution of one-dimensional constant-coefficient advection-diffusion equations, in which piecewise-linear trial and test functions are used in the weak formulation. The strong potential of this method lead to the development of multidimensional ELLAM schemes using either the finite element or finite volume approaches [4,9,13,18–21]. Most of the work on ELLAM methods followed the traditional approach in which first- and second-order numerical methods are probably the most widely used in the numerical simulation to subsurface porous medium flow processes, especially in terms of characteristic methods. The main reasons for the choice of lower-order methods is the fact that most mathematical models for subsurface porous medium flows admit solutions with moving steep fronts and relatively complicated structures. As a result, the solutions of these models usually have minimal regularities. This is due to a number of factors including the strong heterogeneity of the porous media, the non linearity and close couplings between the equations in the system, the strong effect of the singular sources and sinks, the compressibility of the fluid mixture and the medium, and the enormous size of field-scale application and the required long time period of prediction. In these circumstances, higher-order numerical schemes usually cannot reach higher-order convergence rates. Nevertheless, it is well known that higher-order numerical methods have generated numerical solutions with greatly improved accuracy and resolution in the context of nonlinear hyperbolic conservation laws, despite the fact that the solutions to nonlinear conservation laws exhibit shock discontinuities, complicated structures, and minimal regularities.

Motivated by the success of higher-order methods in aerodynamics and other applications, we continue our research on the development of higher-order ELLAM schemes for the solution of transient advection-diffusion equations. We have worked on the development of higher-order-in-time ELLAM schemes with success [1,16]. We have also considered higher-order in space ELLAM schemes in one-space dimension [2]. In this paper, we present an ELLAM scheme, for the transient advection-diffusion equation in two space dimensions, which uses higher-order bi-quadratic trial and test functions in space. The derived scheme possesses a higher-order spatial convergence rate compared to previous two-dimensional ELLAM methods and improved spatial resolution. Numerical experiments are presented to illustrate the performance of the new ELLAM scheme and to numerically verify its convergence rates, which were derived theoretically for one-dimensional problems [17].

The paper is organized in the following way. In the next section, we review the development of the ELLAM method. The details of the numerical approximation will be discussed in Section 3. Numerical results will be reported in Section 4. Some relevant discussions will be presented in the final section.