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WEAK-DUALITY BASED ADAPTIVE FINITE ELEMENT METHODS FOR PDE-CONSTRAINED OPTIMIZATION WITH POINTWISE GRADIENT STATE-CONSTRAINTS*

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Abstract

Adaptive finite element methods for optimization problems for second order linear elliptic partial differential equations subject to pointwise constraints on the ℓ^2-norm of the gradient of the state are considered. In a weak duality setting, i.e. without assuming a constraint qualification such as the existence of a Slater point, residual based a posteriori error estimators are derived. To overcome the lack in constraint qualification on the continuous level, the weak Fenchel dual is utilized. Several numerical tests illustrate the performance of the proposed error estimators.

Mathematics subject classification: 65N30, 90C46, 65N50, 49K20, 49N15, 65K10.
Key words: Adaptive finite element method, A posteriori errors, Dualization, Low regularity, Pointwise gradient constraints, State constraints, Weak solutions.

1. Introduction

In this paper we study the state constrained optimal control problem

$$\left\{\begin{array}{ll} \text{minimize} & J(y,u) := \frac{1}{2}\|y-y_d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u\|_U^2 \quad \text{over } (y,u)\in V\times U \\ \text{subject to} & Ay = u+f \text{ in } V^*, \\ & |\nabla y| \le \psi \text{ a.e. in } \Omega, \end{array}\right. \tag{P}$$

where $\Omega\subset\mathbb{R}^n$, $n\in\{1,2,3\}$, is an open, bounded domain with boundary $\Gamma:=\partial\Omega$, $V:=H_0^1(\Omega)$, $U=L^2(\Omega)$, $y_d\in L^2(\Omega)$, $\alpha>0$, $A:V\to V^*$ denotes a self-adjoint second order linear elliptic partial differential operator, $f\in L^2(\Omega)$, and $\psi\in L^2(\Omega)$ with $\psi\ge\underline{\psi}$ a.e. in Ω for some $\underline{\psi}\in\mathbb{R}_{++}$. Here and below $\|\cdot\|_{0,\Omega}$ refers to the standard $L^2(\Omega)$-norm. In (P) we have $\|\cdot\|_U=\|\cdot\|_{0,\Omega}$. We call y the state and u the control. Of course, more general objective functionals are conceivable, but our choice reflects the often considered tracking-type objective involving a desired state y_d, which may result from measurements, and control costs α. Moreover, convex quadratic

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objectives and affine partial differential equation (PDE) constraints such as those in (P) appear naturally in sequential quadratic programming approaches in optimization.

Pointwise constraints on the gradient of the state, as imposed in (P), are important, e.g., in material science in order to avoid large material stresses. Such stresses may arise from unbalanced cooling regimes in transient phenomena and/or due to the geometric structure of the underlying PDE domain. Usually large stresses cause adverse effects in the material leading to reduced life time or other deterioration. Geometric features of the PDE domain such as cracks or re-entrant corners play also a crucial role in stationary cases in elasticity, where stresses are usually high at the crack tip or the re-entrant corner of, e.g., an L-shaped domain. Thus, it might be desirable to exercise some control in order to reduce these potentially adverse effects.

Even in situations where the PDE domain is smooth, from an optimization theoretic point of view pointwise constraints on the state lead to poor multiplier regularity when characterizing first order optimality by means of a Karush-Kuhn-Tucker (KKT) system. Corresponding theoretical studies can be found in [7–9]. For the derivation of such a KKT-system it is commonly invoked that the feasible set of the optimization problem (like (P)) admits a so-called Slater point. In connection with (P), this requirement results in a function space setting of $V := W^{2,r}(\Omega) \cap H_0^1(\Omega)$, with $r > n$, for the state space and $U := L^r(\Omega)$ for the control space. This yields $\nabla y \in C(\bar{\Omega})^n$ which is needed for the existence of a Slater point.

For pointwise constraints on the gradient and less regular domains (like cracked domains or L-shapes) such a high regularity of the state is out of reach [14]. Hence, the derivation of a primal-dual first order optimality characterization cannot rely on standard tools requiring a constraint qualification such as the existence of a Slater point. As a consequence, one may need to work under a weaker first order condition, i.e., without a bounded set of multipliers associated with $|\nabla y| \leq \psi$ a.e. in Ω. We also note that the domain and the bound ψ have to be compatible in order to yield a non-empty feasible set of (P). For example, requiring $\psi \in L^\infty(\Omega)$ in the presence of a crack, which, however, rules out L^∞-regularity of the gradient of the solution of our PDE with $L^2(\Omega)$-right-hand side, causes an incompatibility and, thus, an empty feasible set. In such cases the optimization problem (P) is void. Hence, throughout this paper we assume that such a data compatibility holds true, i.e., we may assign a well-defined solution operator $G : V^* \to V$ to the PDE in (P).

Adaptive finite element methods have been widely and successfully used for the efficient numerical solution of boundary and initial-boundary value problems for partial differential equations; see, e.g., the monographs [1,3,4,13,23,26] and the many references therein. Recently, residual based a posteriori as well as dual-weighted residual based goal oriented estimators for PDE-constrained optimization problems with pointwise constraints on the control or the state were studied; see, e.g., [5, 17–19, 21, 22, 27]. Concerning constraints on the gradient of the state, however, the present literature is rather scarce; here we refer to recent a priori estimates in [11] based on a certain mixed finite element approach, and to [15]. Compared to pointwise constraints on the state, i.e., $y \leq \Psi$ a.e. in Ω, gradient constraints involve the gradient operator, which has a non-trivial kernel, and require very smooth, i.e. $C^1(\bar{\Omega})$, states in order to guarantee a constraint qualification, such as the existence of a Slater point. Both aspects trouble the existence of a bounded set of Lagrange multiplier with the latter preventing practically relevant, non-smooth PDE domains, as pointed out above. This also has an immediate effect in the a posteriori error analysis as one has to avoid explicit use of a Lagrange multiplier.

In the present paper we are, thus, interested in developing reliable residual based a posteriori error estimators for an adaptive finite element discretization of (P). In particular we study the