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SUPERGEOMETRIC CONVERGENCE OF SPECTRAL COLLOCATION METHODS FOR WEAKLY SINGULAR VOLTERRA AND FREDHOLM INTEGRAL EQUATIONS WITH SMOOTH SOLUTIONS*

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Abstract

A spectral collocation method is proposed to solve Volterra or Fredholm integral equations with weakly singular kernels and corresponding integro-differential equations by virtue of some identities. For a class of functions that satisfy certain regularity conditions on a bounded domain, we obtain geometric or supergeometric convergence rate for both types of equations. Numerical results confirm our theoretical analysis.

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Key words: Weakly singular kernel, Integro-differential equations, Collocation method.

1. Introduction

Given $0 < \mu < 1$, we consider two classes of linear Volterra type integral equations of the form

$$y(t) - \int_0^t (t-s)^{-\mu} y(s)ds = b(t), \qquad t \in (0,T], \tag{1.1}$$

and of the form (Fredholm)

$$y(t) - \int_0^T |t-s|^{-\mu} y(s)ds = b(t), \qquad t \in (0,T], \tag{1.2}$$

where $y(t)$ is the unknown function and $b(t)$ is a sufficiently smooth function. We also consider the corresponding integro-differential equations

$$y'(t) = a(t)y(t) + \int_0^t (t-s)^{-\mu} y(s)ds + b(t), \quad y(0) = y_0, \quad t \in (0,T], \tag{1.3}$$

and

$$y'(t) = a(t)y(t) + \int_0^T |t-s|^{-\mu} y(s)ds + b(t), \quad y(0) = y_0, \quad t \in (0,T], \tag{1.4}$$

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where $y(t)$ is the unknown function, $a(t)$ is an analytic function, and $b(t)$ is a sufficiently smooth function. We assume that each of these equations possesses a unique solution [9, 21].

Numerical approximation of integral equations with singular kernel has caught numerous attentions, among which extensive studies can be found in [1, 10] while numerical solutions of integro-differential equation are studied in [2, 12, 18] etc. Many numerical analysts used graded meshes to develop numerical schemes with an optimal order of convergence, see, e.g., [1,9,11,12]. A hybrid collocation method was also proposed to solve Volterra equations with weakly singular kernels [5] and Fredholm singular equations [3,4]. Recently, in [6,7] a spectral Jacobi-collocation method was proposed and analyzed to solve Volterra equations. The basic idea is to collocate equations at some Jacobi points and use a highly accurate quadrature to approximate the integration in (1.1). In this article, however, instead of numerical integration, we apply exact integration to the composition of the Legendre polynomials and the weakly singular kernel. The exact integration leads to a more accurate solution and reduces the computation cost. It will be shown that a geometric (supergeometric) rate of convergence can be achieved by using our method if $y(t)$ satisfies condition (R): $\|y^{(k)}\|_{L^\infty[0,T]} \leq Ck!R^{-k}$ (condition (M): $\|y^{(k)}\|_{L^\infty[0,T]} \leq CM^k, M > 0$) not only for Volterra equations but also for Fredholm equations as well as their corresponding integro-differential equations. Here, R is sufficiently large. If R is small, an *hp*-version of our method is necessary. For a Volterra equation, if the solution is not smooth enough we may take some function transformations as in [7] to obtain a new equation which possesses better regularity.

In the traditional analysis of spectral methods, the error bound is given in the form $\mathcal{O}(p^{-k})$ for any positive integer k, where p is the polynomial degree. This is to say that the convergence is superior to any polynomial rate as long as the exact solution is in C^∞. Nevertheless, it is still not geometric convergence in its precise sense. The fundamental difference of our analysis here is that we establish a convergence rate in the form $\mathcal{O}(e^{-\sigma p}) = \mathcal{O}(R^{-p})$ (geometric convergence) or $\mathcal{O}(e^{-\sigma p(\ln p - \ln \gamma)}) = \mathcal{O}((\gamma/p)^{\sigma p})$ (super-geometric convergence). For the former, we need condition (R) on the exact solution, and for the later we need condition (M), a more restricted assumption.

Let us elaborate some more on those two conditions. Actually, condition (R) is the analytic assumption. Based on the Cauchy integral formulae, an analytic function f, and its kth derivative $f^{(k)}$, can be expressed as,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw, \quad f^{(k)}(z) = \frac{k!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{k+1}}dw,$$

in its analytic region D with boundary Γ. Condition (M) characterizes a class of entire functions. Typical candidates are $\sin Mz, \cos Mz, e^{Mz}$ and their combinations. For more details, see the discussion in [20].

We see that both conditions are very restricted and satisfied only in special situations, such as when the singular kernel is replaced by a smooth kernel $k(t, s)$ (or $\mu = 0$) and input data a, b are analytic functions. Nevertheless, our numerical schemes is still valid for problems with the singular kernel discussed in this paper, even though the associated analysis for solutions with singularity is lacking. We would emphasis that the spectral method is particularly efficient for equations with sufficiently smooth solutions. Regarding supergeometric convergence of spectral collocation method for differential equations, readers are referred to [19, 20].

This paper is organized as follows: In Sections 2, some preliminary knowledge as well as algorithms for both types of integral equations are given. In Section 3, convergence analysis