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EFFECTS OF INTEGRATIONS AND ADAPTIVITY FOR THE EULERIAN–LAGRANGIAN METHOD*

Jiwei Jia
LMAM & School of Mathematical Science, Peking University, Beijing 100871, China
Email: jiajiwei@math.pku.edu.cn
Xiaozhe Hu
Beijing International Center for Mathematical Research, Beijing 100871, China
Email: huxiaozhezju@gmail.com
Jinchao Xu
Department of Mathematics, Pennsylvania State University, University Park, PA, 16802, USA
Email: xu@math.psu.edu
Chen-Song Zhang
School of Mathematical Sciences, Peking University, Beijing 100871, China
Email: zhangchensong@gmail.com

Abstract

This paper provides an analysis on the effects of exact and inexact integrations on stability, convergence, numerical diffusion, and numerical oscillations for the Eulerian–Lagrangian method (ELM). In the finite element ELM, when more accurate integrations are used for the right-hand-side, less numerical diffusion is introduced and better approximation is obtained. When linear interpolation is used for numerical integrations, the resulting ELM is shown to be unconditionally stable and of first-order accuracy. When Gauss quadrature is used, conditional stability and second-order accuracy are established under some mild constraints for the convection-diffusion problems. Finally, numerical experiments demonstrate that more accurate integrations lead to better approximation, and spatial adaptivity can substantially reduce numerical oscillations and smearing that often occur in the ELM when inexact numerical integrations are used.

Mathematics subject classification: 65M25, 65M60.
Key words: Convection-diffusion problems; Eulerian–Lagrangian method; Adaptive mesh refinement.

1. Introduction

In many physical problems, convection dominates diffusion; for these nearly hyperbolic problems, classical Galerkin finite element methods may suffer from instability and it is natural to explore the method of characteristics (MoC). It is well-known, however, that deformation of mesh in the pure Lagrangian framework could lead to deterioration of accuracy of the numerical solution. The finite element Eulerian–Lagrangian method (ELM) [1, 2] seeks the position of a particle at previous time that reaches a certain point at current time. Thus, the diffusion operator is always solved on a fixed mesh, eliminating the need for mesh regeneration. This method has many variants (see [3, 4] and references therein); and, it is also known as the semi-Lagrangian method (SLM) in the meteorological community (cf. [5]).

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The ELM has several desirable features: (a) it allows relatively large time-step size; (b) it results in a symmetric positive definite discrete linear system, which allows usage of available optimal iterative solvers; (c) it treats (linear or nonlinear) convection terms in a uniform way and the nonlinearity can be handled by solving various ordinary differential equations (ODEs), which can be done in parallel easily.

Despite of all these advantages, the ELM is known to have several disadvantages: (1) it is sensitive to the accuracy of integration/interpolation; (2) it can introduce certain level of numerical diffusion in practice; (3) its computational overhead for back tracking is usually heavy. We refer to [6–8] and the reference therein for more details. Indeed, (1) and (2) are essentially related. Usually, the finite difference ELMs are more diffusive when first-order interpolation schemes are employed; this is similar to the upwinding scheme [9]. In the finite element framework, numerical diffusion is much smaller. We can expect that numerical diffusion can be reduced when more accurate integrations are used on the right-hand-side of finite element weak formulation. In particular, for a simple one-dimensional case, we prove that the ELM is free of numerical diffusion when the right-hand-side is integrated exactly; see Section 3. For more general cases, our numerical experiments also confirms this expectation; see Section 6.

There have been discussions on the convergence and stability of the ELM; we refer to [1, 2, 10, 11] for a priori error estimations, and [12, 13] for studies on numerical stability. Most analysis on this method in the literature have been carried out under the assumption that all integrations are evaluated exactly. However, numerical quadratures often have to be used to evaluate these integrations for two- and three-dimensional problems in practice. Without the exact-integration assumption, analysis of the ELM is much more involved. To the best of our knowledge, [12] and [13] are the only papers that analyzed the effect of numerical quadratures theoretically. More specifically, [12] studied how the stability of the ELM is compromised by some classical quadrature rules in one dimensional case for pure transport problems.

A posteriori error analysis and spatial mesh adaptivity have been applied to the ELM; see [14–17]. In particular, [16] gave a residual-based $L^2(L^2)$ a posteriori error estimator; however, the numerical experiments therein indicated that the norm of the residual on an individual element may be a poor estimate of the local error (the norm of the residual can be used to bound the error on a global basis from above.) In [17], the authors derived a sharp $L^\infty(L^1)$ a posteriori error estimator for a nonlinear convection-diffusion equation, which is discretized with the ELM implicitly in time and the continuous piecewise linear finite element in space. We will use the spatial error estimators proposed in [17] to drive out adaptive mesh refinements in Sections 5 and 6.

In this paper, we make the following observations on ELM with inexact numerical integrations through theoretical analysis and numerical experiments:

- *Numerical diffusion and dispersion.* We observe that, when exact integration is employed, very little numerical diffusion and dispersion are introduced by ELM. On the other hand, ELM with linear interpolations tends to introduce excessive numerical diffusion.

- *Stability and convergence rate.* We show the conditional stability and optimal convergence rate for ELM with Gauss quadratures, and unconditional stability and suboptimal convergence rate for ELM with linear interpolations.

- *Effects of adaptive mesh refinement.* The adaptive mesh refinement can not only stabilize the scheme but also reduce numerical diffusion.