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HIGH ORDER COMPACT FINITE DIFFERENCE SCHEMES FOR THE HELMHOLTZ EQUATION WITH DISCONTINUOUS COEFFICIENTS*

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Abstract

In this paper, third- and fourth-order compact finite difference schemes are proposed for solving Helmholtz equations with discontinuous media along straight interfaces in two space dimensions. To keep the compactness of the finite difference schemes and get global high order schemes, even at the interface where the wave number is discontinuous, the idea of the immersed interface method is employed. Numerical experiments are included to confirm the efficiency and accuracy of the proposed methods.

Mathematics subject classification: 65N06.
Key words: Helmholtz equation, Compact finite difference scheme, Discontinuous media, Immersed interface method, Nine-point stencil.

1. Introduction

In this paper, we consider the following two-dimensional Helmholtz equation

$$\Delta u + k_0^2 \nu(x) u = f(x,y), \quad \text{or} \quad \Delta u + k^2 u = f(x,y), \quad (x,y) \in \Omega \tag{1.1}$$

in a rectangular domain Ω with a Dirichlet boundary condition, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. The material coefficient $\nu(x)$ is assumed to be piecewise constant and has a finite jump across a straight line $\Gamma = \{(x,y), x = x_0\}$ in the domain, see [7] for a reference and applications of such a problem. For convenience, we will use the notation $k^2 = k_0^2 \nu(x)$ which is piecewise constant. Across the interface, the solution satisfies the following natural jump conditions

$$[u] = 0, \qquad [u_x] = 0, \qquad [u_y] = 0. \tag{1.2}$$

Notice that the second- or higher-order partial derivatives with respect to x, and the source term $f(x,y)$ may be discontinuous across the interface, see Fig. 1.1 for an illustration.

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Helmholtz equations describe many physical phenomena, including acoustic, elastic, and electromagnetic waves. Standard numerical methods, such as the boundary element method [16], finite element methods [12, 19], and finite difference methods [25, 26], have been employed to solve the Helmholtz equation. Applications of the Helmholtz equations with discontinuous media can be found, for example, [7, 15].

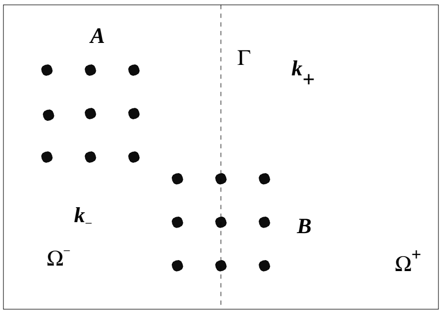

Fig. 1.1. A diagram of the problem and the finite difference stencil. A represents a regular finite difference stencil, while B represents an irregular finite difference stencil that involves grid points from both sides of the interface Γ.

The main difficulty in solving Helmholtz equations is that the solution is highly oscillatory for high wave number. The phase error (pollution) of the computed solution obtained with low order discretization is large unless fine meshes are used per wavelength, see for example [12]. A fine mesh would lead to a large system of equations which may be computationally prohibitive. Many different approaches have been proposed to reduce the phase error. For example, the high-order finite element method was proposed in [10]; the h-version and h-p-version finite element methods were proposed in [13, 14]. In [19], a standard bilinear finite element together with a modified quadrature rule were used, which led to fourth-order phase accuracy on orthogonal uniform meshes for a constant wave number. Recently, high-order accurate methods, such as spectral methods and high order finite difference schemes, have been successfully developed in solving the Helmholtz equation when k is constant; see e.g., [2, 3, 17, 18]. But when k is a piecewise constant, it becomes more difficult to get high order methods. In this case, the solution and its first-order derivative are continuous everywhere [15], whereas the second-order and higher derivatives or the right hand side $f(x, y)$ can have jumps at the discontinuity of k. In [6], Gustafsson and Wahlund analyzed the effect of discontinuous coefficients on the phase and amplitude errors. It was shown that the schemes in [5] were only first-order accurate even if they are second- or fourth-order accurate for smooth solutions. In [7], Baruch et. al. constructed high order finite volume schemes for the one-dimensional Helmholtz equation with discontinuous coefficients based on the integral form of the Helmholtz equation. Their schemes can keep global higher-order accuracy in the presence of discontinuities in the coefficients.

In this paper, we propose compact high-order (third and fourth) finite difference schemes to solve the two-dimensional Helmholtz equation with piecewise constant wave numbers. The compact high-order schemes are developed using the continuation of solutions by the Taylor series expansion and the immersed interface method, see for example, [23, 24]. Our high order finite difference schemes are based on the centered nine-point stencil, hence, the scheme is called compact, see [1] for the definition. The third-order compact scheme developed in this paper is simple and easy to derive. The fourth-order compact scheme may be necessary if the wave number is relatively large. Compact schemes have been developed for a variety of