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EFFECTS OF APPROXIMATE DECONVOLUTION MODELS ON THE SOLUTION OF THE STOCHASTIC NAVIER-STOKES EQUATIONS*

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Abstract

The direct numerical simulation of Navier-Stokes equations in the turbulent regime is not computationally feasible either in the deterministic or (especially) in the stochastic case. Therefore, turbulent modeling must be employed. We consider the family of approximate deconvolution models (ADM) for the simulation of the turbulent stochastic Navier-Stokes equations (NSE). For moderate values of the Reynolds number, we investigate the effect stochastic forcing (through the boundary conditions) has on the accuracy of solutions of the ADM equations compared to direct numerical simulations. Although the existence, uniqueness and verifiability of the ADM solutions has already been proven in the deterministic setting, the analyticity of a solution of the stochastic NSE is difficult to prove. Hence, we approach the problem from the computational point of view. A Smolyak-type sparse grid stochastic collocation method is employed for the approximation of first two statistical moments of the solution − the expected value and variance. We show that for different test problems, the modeling error in the stochastic case is the same as predicted for the deterministic setting. Although the ADMs are arguably only applicable for certain boundary conditions (zero or periodic), we test the model on a problem with a boundary layer and recirculation region and demonstrate that the model correctly predicts the solution of the stochastic NSE with the noise in the boundary data.

Mathematics subject classification: 65M70, 76F65.
Key words: Turbulence modeling, Stochastic Navier-Stokes equations, Deconvolution.

1. Introduction

Realistic simulations of complex systems governed by nonlinear partial differential equations (in this paper we consider the case of fluid flow, described by Navier-Stokes equations (NSE)) must account for "noisy" features of modeled phenomena, such as material properties, coefficients, domain geometry, excitations and boundary data. "Noise" can be understood as uncertainties in the specification of the physical model. In an attempt to capture the noisy aspects of the system, we describe the input data on the boundary as random fields. In this work we consider the boundary data that can be described by a finite number of random variables.

Direct numerical simulation of a turbulent flow is often not computationally economical or even feasible. The problem is magnified many times over in the case of the stochastic NSE because deterministic sampling must be employed; one solves the discrete turbulent Navier-Stokes system with random coefficients many times, once for each sampling of the random data. Although these computations can be parallelized, the direct simulation in the turbulent

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case is not feasible in the foreseeable future. One has to introduce a turbulent model, as is the case for the deterministic turbulent NSE.

The largest structures in the flow (containing most of the energy) are responsible for much of the mixing and most of the momentum transport. This observation led to the development of various numerical regularizations; one of these is Large Eddy Simulation (LES) that is based on the idea that the flow can be represented by a collection of scales with different sizes and, instead of trying to approximate all of them down to the smallest one, one defines a filter width $\delta > 0$ and computes only the scales of size bigger than δ (large scales) whereas the effect of the small scales on the large scales is modeled. This reduces the number of degrees of freedom in a simulation and represents accurately the large structures in the flow.[1] In this report we consider one particular LES model, the Approximate Deconvolution Model (ADM), introduced in [1]. The model has been extensively studied in the deterministic setting; see, e.g., [11,13,15,18] and the references therein.

The ADM for the stochastic NSE with the noise on the boundary is given by

$$w_t - \frac{1}{\mathrm{Re}}\Delta w + \nabla \cdot \overline{(G_N w)(G_N w)}^{\delta} + \nabla q = \overline{f}^{\delta}, \tag{1.1a}$$

$$\nabla \cdot w = 0, \tag{1.1b}$$

subject to

$$w(0,x,\omega) = \overline{u}_0^{\delta}(x)$$

and noisy boundary conditions

$$w(t,x,\omega)|_{\partial\Omega} = \overline{u}^{\delta}(t,x)|_{\partial\Omega} + \sum_{i=1}^{K} \omega_i \Phi_i.$$

Here G_N is an approximate deconvolution operator, defined in Section 2. For the computational tests we consider the zeroth order ADM with $N = 0$.

The solution of (1.1) therefore depends on K random variables. In the computational tests we use $K = 2$ to reduce the computational cost. We assume $\Gamma_k = [-1,1]$, $\forall k = 1, \cdots, K$, where Γ_k denotes the image of k-th random variable. We let $\Gamma^K = \prod_{k=1}^{K} \Gamma_k$; assume also that the random variables have a joint probability density function

$$\rho : \Gamma^K \to \mathbb{R}_+, \ \text{with } \rho \in L^{\infty}(\Gamma^K).$$

For all the test problems we will assume that the given probability density functions are uniform.

Even though the ADM solution of the deterministic NSE is computationally feasible, the most popular approach to solving a partial differential equation in a probabilistic setting (the Monte Carlo method) is too costly due to a large number of sampling. Hence, in order to obtain a solution to a stochastic turbulent NSE, we need to combine the ADM turbulence model with a probabilistic method which has higher convergence rate than the Monte Carlo method.

Different methods have been proposed for solving probabilistic partial differential equations with (in certain cases) a much higher convergence rate than the Monte Carlo method. We

[1] One should notice that there is always a dilemma when choosing the filtering width δ. The larger δ is, the less costly the computations are (less degrees of freedom left), but the larger the modeling error is. On the other hand, choosing δ too small makes the problem too computationally costly. Usually δ is taken to be of order h, the diameter of the mesh employed. This guarantees that the computations are feasible and at the same time we only need to model the eddies of the size smaller than the mesh diameter.