Journal of Computational Mathematics
Vol.29, No.1, 2011, 91–107.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1006-m2991

A PRIORI ERROR ESTIMATES OF A COMBINED MIXED FINITE ELEMENT AND DISCONTINUOUS GALERKIN METHOD FOR COMPRESSIBLE MISCIBLE DISPLACEMENT WITH MOLECULAR DIFFUSION AND DISPERSION*

Jiming Yang
College of Science, Hunan Institute of Engineering, Xiangtan 411104, China
Email: yangjiminghnie@163.com
Yanping Chen
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
Email: ypchen@xtu.edu.cn, yanpingchen@scnu.edu.cn

Abstract

A combined approximation for a kind of compressible miscible displacement problems including molecular diffusion and dispersion in porous media is studied. Mixed finite element method is applied to the flow equation, and the transport one is solved by the symmetric interior penalty discontinuous Galerkin method (SIPG). To avoid the inconvenience of the cut-off operator in [3,21], some induction hypotheses different from the ones in [6] are used. Based on interpolation projection properties, a priori hp error estimates are obtained. Comparing with the existing error analysis that only deals with the diffusion case, the current work is more complicated and more significant.

Mathematics subject classification: 65M12, 65M60.
Key words: A priori error, Mixed finite element, Discontinuous Galerkin, Compressible miscible displacement.

1. Introduction

We consider the following single-phase, miscible displacement problem of one compressible fluid by another in porous media:

$$d(c)\frac{\partial p}{\partial t} + \nabla\cdot\boldsymbol{u} = d(c)\frac{\partial p}{\partial t} - \nabla\cdot(a(c)\nabla p) = q, \quad (x,t)\in\Omega\times J, \tag{1.1}$$

$$\phi\frac{\partial c}{\partial t} + b(c)\frac{\partial p}{\partial t} + \boldsymbol{u}\cdot\nabla c - \nabla\cdot(\boldsymbol{D}(\boldsymbol{u})\nabla c) = (\widehat{c}-c)q, \quad (x,t)\in\Omega\times J, \tag{1.2}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad (x,t)\in\partial\Omega\times J, \tag{1.3}$$

$$\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n} = 0, \quad (x,t)\in\partial\Omega\times J, \tag{1.4}$$

$$p(x,0) = p_0(x), \quad x\in\Omega, \tag{1.5}$$

$$c(x,0) = c_0(x), \quad x\in\Omega, \tag{1.6}$$

where Ω is a polygonal and bounded domain in $\mathbb{R}^n$ ($n=1,2$ *or* 3) with boundary $\partial\Omega$, $J=(0,T]$, $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$; $\boldsymbol{u}(x,t)$ represents the Darcy velocity of the mixture and $p(x,t)$ is the fluid pressure in the fluid mixture; $c(x,t)$ is the solvent concentration of interested species measured in amount of species per unit volume of the fluid mixture, $\phi(x)$

* Received December 6, 2008 / Revised version received August 24, 2009 / Accepted February 4, 2010 / Published online September 20, 2010 /

is the effective porosity of the medium and is bounded above and below by positive constants, $\boldsymbol{D}(\boldsymbol{u})$ denotes a diffusion or dispersion tensor which has contributions from molecular diffusion and mechanical dispersion. Moreover,

$$\boldsymbol{D}(\boldsymbol{u}) = d_m \boldsymbol{I} + |\boldsymbol{u}|\Big(\alpha_l \boldsymbol{E}(\boldsymbol{u}) + \alpha_t\big(\boldsymbol{I} - \boldsymbol{E}(\boldsymbol{u})\big)\Big),$$

where $\boldsymbol{E}(\boldsymbol{u})$ is the tensor that projects onto the $\boldsymbol{u}$ direction, whose (i,j) component is

$$(\boldsymbol{E}(\boldsymbol{u}))_{i,j} = \frac{u_i u_j}{|\boldsymbol{u}|^2};$$

d_m is the molecular diffusivity and is assumed to be strictly positive; α_l and α_t are the longitudinal and transverse dispersion respectively, and are assumed to be nonnegative. The imposed external total flow rate q is a sum of sources and sinks. That is to say, $q = q^+ + q^-$, where $q^+ = \max(q, 0)$ and $q^- = \min(q, 0)$. q and $\frac{\partial q}{\partial t}$ are assumed to be bounded. The notation $\widehat{c}$ is the specified injected concentration c_w at sources if $q > 0$ and is the resident concentration c at sinks if $q < 0$. We assume that p, ∇p, c and ∇c are essentially bounded.

The coefficients $a(c)$, $b(c)$ and $d(c)$ are defined as:

$$a(c) = \frac{k(x)}{\mu(c)}, \quad b(c) = \phi(x)c_1(z_1 - z_1c_1 - z_2c_2), \quad d(c) = \phi(x)(z_1c_1 + z_2c_2),$$

where $c = c_1 = 1 - c_2$, $\mu(c)$ represents the viscosity, z_j denotes the constant compressibility factor for the jth component $(j = 1, 2)$, $k(x)$ is the permeability of the medium. $a(c)$ and $d(c)$ have positive lower and upper bounds,

$$0 < a_* < a(c) < a^* \ \text{ and } \ 0 < d_* < d(c) < d^*,$$

$b(c)$ is bounded. In addition, $\frac{\partial a(c)}{\partial c}$ is uniformly bounded and Lipschitz continuous with respect to c.

It is well known that the mixed finite element (MFE) method can obtain the same optimal order of convergence for both the pressure and the Darcy velocity and has been widely used in the numerical simulation for porous media problems [8–10].

Recently, M. F. Wheeler, B. Rivière and S. Sun have devoted to using discontinuous Galerkin (DG) solver for problems in porous media [16,20]. V. Dolejsi and M. Feistauer, have investigated DG approximation for convection-diffusion problems (see [7, 12, 13]). DG methods belong to a class of non-conforming methods (see [3, 5, 15, 18, 23–25]) and they solve the differential equations by piecewise polynomial functions over a finite element space without any requirement on inter-element continuity – however, continuity on inter-element boundaries together with boundary conditions is weakly enforced through the bilinear form. DG is very attractive for practical numerical simulations because of its physical and numerical properties. Firstly, it is flexible which allows for general non-conforming meshes with variable degrees of approximation. Secondly, it is locally mass conservative and the average of the trace of the fluxes along an element edge is continuous. Thirdly, it has less numerical diffusion and can deal with rough coefficient problems. Finally, it is easier for the *hp*-adaptivity because the information over cell boundaries is almost decoupled.

To approximate to the exact solution of (1.1)–(1.6), we shall make use of a combined mixed finite element and DG method.

Many scholars have contributed to numerical approximations to miscible displacement problems [4, 14]. Unfortunately, there are very few literature dealing with DG methods. In [21] a