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THE FINITE DIFFERENCE METHOD FOR DISSIPATIVE KLEIN–GORDON–SCHRÖDINGER EQUATIONS IN THREE SPACE DIMENSIONS*

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Abstract

A fully discrete finite difference scheme for dissipative Klein–Gordon–Schrödinger equations in three space dimensions is analyzed. On the basis of a series of the time-uniform priori estimates of the difference solutions and discrete version of Sobolev embedding theorems, the stability of the difference scheme and the error bounds of optimal order for the difference solutions are obtained in $H^2 \times H^2 \times H^1$ over a finite time interval. Moreover, the existence of a maximal attractor is proved for a discrete dynamical system associated with the fully discrete finite difference scheme.

Mathematics subject classification: 65M06, 35Q55, 65P99.
Key words: Dissipative Klein–Gordon–Schrödinger equations, Finite difference method, Error bounds, Maximal attractor.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$, we shall consider a finite difference approximation of dissipative Klein–Gordon–Schrödinger (KGS) equations [1]

$$i\psi_t + \Delta\psi + i\alpha\psi + \phi\psi = F, \qquad in \ \ \Omega, \ t > 0 \tag{1.1a}$$

$$\phi_{tt} + \beta\phi_t - \Delta\phi + \mu^2\phi = |\psi|^2 + G, \quad in \ \ \Omega, \ t > 0 \tag{1.1b}$$

with boundary condition

$$(\psi, \phi)|_{\partial\Omega} = 0, \quad t > 0, \tag{1.2}$$

and initial conditions

$$(\psi, \phi, \phi_t)(x, 0) = (\psi_0, \phi_0, \phi_1)(x), \quad in \ \ \Omega, \tag{1.3}$$

where ψ and ϕ represent a complex scalar neucleon field and a real meson field respectively, α, β and μ^2 are positive constants, F and G are given complex and real functions, respectively.

It is convenient to reduce (1.1) to an evolution equation of the first order in time. For this purpose, let $\varepsilon > 0$ be a fixed constant, satifying $\varepsilon \le \min\big(\beta/2, \mu^2/\beta\big)$. We introduce $\theta = \phi_t + \varepsilon\phi$.

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Then the problem (1.1)–(1.3) is equivalent to the following problem

$$i\psi_t + \Delta\psi + i\alpha\psi + \phi\psi = F, \quad in \;\; \Omega, \; t > 0, \tag{1.4a}$$

$$\phi_t + \varepsilon\phi - \theta = 0, \quad in \;\; \Omega, \; t > 0, \tag{1.4b}$$

$$\theta_t + (\beta - \varepsilon)\theta + \Big(\mu^2 - \varepsilon(\beta - \varepsilon) - \Delta\Big)\phi = |\psi|^2 + G, \quad in \;\; \Omega, \; t > 0, \tag{1.4c}$$

with boundary conditions

$$(\psi, \phi, \theta)|_{\partial\Omega} = 0, \quad t > 0, \tag{1.5}$$

and initial conditions

$$(\psi, \phi, \theta)(x, 0) = (\psi_0, \phi_0, \theta_0)(x), \quad in \;\; \Omega. \tag{1.6}$$

In the conservative case, i.e., $\alpha = \beta = 0$, $F = G = 0$, the system has been studied by many authors, see, e.g., [2, 3, 5, 13] and so on. In the dissipative case ($\alpha > 0, \beta > 0$), the long time behavior of infinite dimensional dynamical system $S(t)$ associated with the initial boundary value problem (1.1)–(1.3) has been studied in [1, 4, 9]. Biler [1] proved that the maximal attractor $\mathcal{A}$ exists in the weak topology of $H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$, which has finite Hausdorff and fractal dimension. Li [9] proved the existence of finite dimensional maximal attractor in the topology of $H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

At the same time, in numerical simulation of the continuous dynamical system, we are interested in study of the dynamical properties of the discrete dynamical system associated with the numerical scheme for problem (1.1). It is important that the discrete dynamical system can remain some properties of the continuous dynamical system such as dissipatedness. It is our purpose in this paper to consider a discrete dynamical system associated with the fully discrete finite difference scheme for problem (1.4)–(1.6). It will be proved that for each mesh size, the discrete dynamical system also possesses a maximal attractor. A similar problem was studied by many authors, see, e.g., [7, 11, 12, 14, 15].

The rest of this paper is organized as follows. After introducing some notations, in Sect. 2 we give several embedding theorems and interpolation inequalities for discrete functions, which are the analogues of embedding theorems and interpolation inequalities for the Sobolev space $W^{m,p}(\Omega)$. In Sect. 3, a fully discrete finite difference scheme is established for problem (1.4) with the homogeneous Dirichlet boundary condition (1.5). The existence of the solutions of the fully discrete finite difference scheme is proved by using the Leray-Schauder fixed point theorem. Then we establish some uniform bounds of the solutions in suitable norms. In Sect. 4, we obtain the stability and the convergence properties for the finite difference scheme over a finite time interval $(0, T]$. Finally, in Sect. 5, by regarding the fully discrete finite difference scheme as a discrete dynamical system $S_{h,\Delta t}(t_n)$ that is an approximation of the dynamical system $S(t)$, and by using the results in Sections 3 and 4, we prove the existence of an absorbing set and an attractor for the discrete dynamical system $S_{h,\Delta t}(t_n)$.

2. Some Notations and Lemmas

Assume that the domain Ω is the three-dimensional rectangular domain $(0, l_1) \times (0, l_2) \times (0, l_3)$, where l_i $(i = 1, 2, 3)$ are positive constants. Let us divide the domain $\overline{\Omega}$ into small grids by the parallel planes $x = ih_1$ $(0 \le i \le J_1), y = jh_2$ $(0 \le j \le J_2)$ and $z = kh_3$ $(0 \le k \le J_3)$, where h_1, h_2, h_3 are the spatial mesh lengths, J_1, J_2, J_3 are positive integers, and $J_i h_i = l_i$ $(i = 1, 2, 3)$. Let $\psi_h, \phi_h, u_h, v_h, \cdots$ denote complex-valued or real-valued discrete functions