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ON BLOCK MATRICES ASSOCIATED WITH DISCRETE TRIGONOMETRIC TRANSFORMS AND THEIR USE IN THE THEORY OF WAVE PROPAGATION*

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Abstract

Block matrices associated with discrete Trigonometric transforms (DTT's) arise in the mathematical modelling of several applications of wave propagation theory including discretizations of scatterers and radiators with the Method of Moments, the Boundary Element Method, and the Method of Auxiliary Sources. The DTT's are represented by the Fourier, Hartley, Cosine, and Sine matrices, which are unitary and offer simultaneous diagonalizations of specific matrix algebras. The main tool for the investigation of the aforementioned wave applications is the efficient inversion of such types of block matrices. To this direction, in this paper we develop an efficient algorithm for the inversion of matrices with U-diagonalizable blocks (U a fixed unitary matrix) by utilizing the U-diagonalization of each block and subsequently a similarity transformation procedure. We determine the developed method's computational complexity and point out its high efficiency compared to standard inversion techniques. An implementation of the algorithm in Matlab is given. Several numerical results are presented demonstrating the CPU-time efficiency and accuracy for ill-conditioned matrices of the method. The investigated matrices stem from real-world wave propagation applications.

Mathematics subject classification: 65F05, 65T50, 74J20, 78A40, 15A09.
Key words: Discrete Trigonometric transforms, Block matrices, Efficient inversion algorithms, Wave radiation and scattering, Numerical methods in wave propagation theory.

1. Introduction

Discrete Trigonometric transforms (DTT's) play a significant role in wave scattering and radiation theory, signal processing, physics, and numerical linear algebra. Representative examples constitute the discrete Fourier transforms (DFT's), the discrete Hartley transforms (DHT's), the discrete Cosine transforms (DCT's), and the discrete Sine transforms (DST's). Their primary contribution lies in the significant reduction of the complexity in the associated mathematical problems. For example, applications of such appropriate transforms in differential and integral equations reduce them to algebraic equations, whose solutions are more easily obtained, see, e.g., [1, 2]. Moreover, in harmonic analysis as well as in signal processing the DFT decomposes a signal sequence into its frequency components [3]. It is important to note that this wide applicability of the DTT's is mainly justified by the existence of fast algorithms, that allow the transforms computations within $\mathcal{O}(n\log_2 n)$ (instead of $\mathcal{O}(n^2)$ when performing directly the matrix-vector product of length n) [4–6].

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In particular, block matrices associated with DTT's arise in concrete physical and technological applications including: (i) the solution of wave scattering and radiation problems with the Method of Moments (MoM) [7, 8], (ii) the investigation and optimization of numerical methods for electromagnetic scattering problems, such as the Method of Auxiliary Sources (MAS) [9]- [10], (iii) the numerical solution of integral equations with the Boundary Element Method (BEM) [11], (iv) optical imaging [12], (v) image compression [13], (vi) efficient preconditioning of Toeplitz systems [14]. Besides, we point out that these applications exhibit the essential role of the inversion of such types of complex block matrices for the derivation of formulas determining the error bounds of numerical methods as well as for the numerical or semi-analytical computation of solutions [15].

The specific DTT's mentioned above are represented by the Fourier, Hartley, Cosine, and Sine matrices. These unitary matrices offer simultaneous diagonalizations of specific matrix algebras, including circulants, skew-circulants, Toeplitz-plus-Hankel, tridiagonal [16]. These matrix algebras are unified by considering the algebra $\mathrm{Diag}(U)$ of all U-diagonalizable matrices, for U a fixed unitary matrix.

For the mathematical modelling of the above mentioned wave applications we develop in this paper an efficient method for the inversion of an $m \times m$ block matrix $A = [A_{ij}]$ with U-diagonalizable blocks of order n $(i, j = 1, \ldots, m)$. First, we consider the diagonalizations $A_{ij} = U\Lambda_{ij}U^*$, where Λ_{ij} is the diagonal matrix containing the eigenvalues of A_{ij}, and hence the inversion of A is reduced to that of the $m \times m$ block matrix $\Lambda = [\Lambda_{ij}]$ with diagonal blocks of order n. For the inversion of Λ we construct, by using concepts of Graph Theory, an appropriate permutation matrix P so that the matrix $P\Lambda P^T = \mathrm{diag}(\Lambda'_1, \Lambda'_2, \ldots, \Lambda'_n)$ is block-diagonal with Λ'_k invertible $m \times m$ full matrices. The inverse $\Lambda^{-1} = [L_{ij}]$ of Λ is then determined by inverting each block Λ'_k with a standard LU direct solver. Finally, the inverse of A is given by the inverse block-diagonalization that is $A^{-1} = [UL_{ij}U^*]$. An implementation in Matlab of the above described algorithmic inversion is given in the Appendix.

For any one of the choices of U, that is Fourier, Hartley, Cosine, and Sine matrices, the matrix multiplications $U^*A_{ij}U$ and $UL_{ij}U^*$, appearing in the block-diagonalizations, are computed by applying the DTT's for $n \neq 2^p$ or the fast Trigonometric transforms (FTT's) for $n=2^p$, that is the fast Fourier, Hartley, Cosine, and Sine transforms [4]- [6]. Hence, the computational complexity (i.e. the total number of required scalar complex multiplications) of the inversion algorithm is $n\mathcal{O}(m^3) + 2m^2n^2$ for $n \neq 2^p$ and $n\mathcal{O}(m^3) + 2m^2\mathcal{O}(n\log_2 n)$ for $n=2^p$. This shows that the developed method is far more efficient than the LU decomposition applied to the original matrix A, having complexity $\mathcal{O}(m^3n^3)$. We note that in several wave applications the order n of each block may be chosen equal to 2^p by selecting suitable discretizations of the scatterer's or radiator's surface [7]- [10].

On the other hand, the above described inversion method can be also applied for the efficient determination of the eigenvalues of a matrix A with U-diagonalizable blocks. Specifically, the eigenvalues of A are those of all blocks Λ'_k and thus their computation requires $n\mathcal{O}(m^3) + m^2\mathcal{O}(n\log_2 n)$ multiplications. Besides, we notice the parallel nature of the proposed inversion algorithm, since the inversion of each specific block Λ'_k can be handled by a different processing unit. For a discussion on parallel algorithms for inverting block matrices which arise in inverse wave scattering theory see [17].

Several numerical results are presented exhibiting the efficiency of the proposed method and highlighting its beneficial contribution in the numerical implementation of certain scattering and radiation applications. We compare in terms of CPU time the developed algorithmic