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WAVE COMPUTATION ON THE HYPERBOLIC DOUBLE DOUGHNUT*

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Abstract

We compute the waves propagating on the compact surface of constant negative curvature and genus 2 that is a toy model in quantum chaos theory and cosmic topology. We adopt a variational approach using finite elements. We have to implement the action of the fuchsian group by suitable boundary conditions of periodic type. Despite the ergodicity of the dynamics that is quantum weak mixing, the computation is very accurate. A spectral analysis of the transient waves allows to compute the spectrum and the eigenfunctions of the Laplace-Beltrami operator. We test the exponential decay due to a localized dumping satisfying the assumption of geometric control.

Mathematics subject classification: 65M, 65N, 58J, 37D.
Key words: Wave equation, Hyperbolic manifold, Finite elements, Quantum chaos.

1. Introduction

The Hyperbolic Double Doughnut $\mathbf{K}$ is the compact surface of negative constant curvature with two holes. We can define it by the quotient of the hyperbolic Poincaré disc $\mathbf{D}$, by some Fuchsian group Γ. Alternatively, we can construct it as the quotient of the so-called Dirichlet polygon, or fundamental domain $\mathcal{F} \subset \mathbf{D}$, by a suitable relation of equivalence $\sim$:

$$\mathbf{K} = \mathbf{D}/\Gamma = \mathcal{F}/\sim .$$

This beautiful object has many fascinating properties as regards the classical and quantum chaos (classical references are [2, 6]). Several important computational investigations of the spectrum were performed by using a stationnary method by Aurich and Steiner [1]. Moreover there has been much recent interest for the cosmological models with non trivial topology (a seminal work is the famous "Cosmic Topology" by Lachièze-Rey and Luminet [8]). In this domain, the fluctuations of the cosmic microwave background (CMB) are investigated and we have to compute the evolution of initial metric perturbations. These cosmological questions consist in solving the wave equation on compact multi-connected hyperbolic manifolds, and computing also the eigenvalues and eigenfunctions of the Laplace-Beltrami operator. In this context, $\mathbf{K}$ has been studied as a toy model in [5], where a scheme based on the finite differences on a euclidean grid was used to solve the D'Alembertian. Nevertheless, such a grid appears to be not convenient for the hyperbolic geometry and unsuitable for a description of $\mathcal{F}$, in particular its boundary. In this paper we compute the solutions of the wave equation in the time domain, by using a variational method and a discretization with finite elements on very fine meshes. The domain of calculus is the Dirichlet polygon, therefore the initial Cauchy problem on the

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manifold without boundary $\mathbf{K}$, becomes a mixed problem on $\mathcal{F}$ and the action of the Fuchsian group is expressed as boundary conditions on $\partial\mathcal{F}$, analogous to periodic conditions. These boundary constraints are implemented in the choice of the basis of finite elements. By this way we obtain very accurate results on the transient waves despite the fact that the dynamics is extremly chaotic (the flow is ergodic and quantum weak mixing [13]). We test these results by performing a Fourier analysis of the transient waves that allows to find a lot of eigenvalues of the Laplace-Beltrami operator $\Delta_{\mathbf{K}}$ on $\mathbf{K}$. We compute also the solutions of the damped wave equation

$$\partial_t^2\psi - \Delta_{\mathbf{K}}\psi + a\partial_t\psi = 0.$$

When $0 \leq a \in L^\infty(\mathbf{K})$ and $a > 0$ on $\partial\mathcal{F}$, the geometric control condition of Rauch and Taylor [11] is satisfied, and our numerical experiments agree with their theoretical results, stating that the energy decays exponentially.

2. The Hyperbolic Double Doughnut

In this part we describe the construction of the Hyperbolic Double Doughnut. First we recall some important properties of the 2-dimensional hyperbolic geometry. It is convenient to use the representation of the hyperbolic space by using the Poincaré disc

$$\mathbf{D} := \left\{(x,y) \in \mathbb{R}^2,\ x^2+y^2<1\right\},$$

endowed with the metric expressed with the polar coordinates by

$$ds^2_{\mathbf{D}} = \frac{4}{(1-r^2)^2}\mathrm{d}r^2 + 4\frac{r^2}{(1-r^2)^2}\mathrm{d}\varphi^2 = \frac{4}{(1-x^2-y^2)^2}[\mathrm{d}x^2+\mathrm{d}y^2].$$

It is useful to use the complex parametrization $z = x+iy$. We have to carefully distinguish the euclidean distance

$$d(z,z') = \mid z - z' \mid,$$

and the hyperbolic distance associated to the hyperbolic metric, given by

$$\cosh d_H(z,z') = 1 + \frac{2|z-z'|^2}{(1-|z|^2)(1-|z'|^2)}.$$

We remark that

$$d(0,z) = \tanh\frac{d_H(0,z)}{2}$$

hence the euclidean circles centered in 0 are hyperbolic circles centered in 0, and more generally, all the hyperbolic circles $\{z';\ \ d_H(z',z_0) = R\}$, with $R > 0$, $z_0 \in \mathbf{D}$, are euclidean circles. The invariant measure $d\mu_H$ on the Poincaré disc allows to compute the area of any Lebesgue measurable subset $X \subset \mathbf{D}$ by the formula

$$\mu_H(X) = \int_X \frac{4}{(1-|z|^2)^2}\,dx\,dy.$$

The group of the isometries of $\mathbf{D}$ is generated by three kinds of transformations.

1. The Rotations of angle $\varphi_0 \in \mathbb{R}$
$$R_{\varphi_0}(z) = e^{i\varphi_0}z,$$