Journal of Computational Mathematics
Vol.28, No.5, 2010, 621–644.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1001-m3006

ADAPTIVE QUADRILATERAL AND HEXAHEDRAL FINITE ELEMENT METHODS WITH HANGING NODES AND CONVERGENCE ANALYSIS*

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Abstract

In this paper we study the convergence of adaptive finite element methods for the general non-affine equivalent quadrilateral and hexahedral elements on 1-irregular meshes with hanging nodes. Based on several basic ingredients, such as quasi-orthogonality, estimator reduction and Döfler marking strategy, convergence of the adaptive finite element methods for the general second-order elliptic partial equations is proved. Our analysis is effective for all conforming $\mathbb{Q}_m$ elements which covers both the two- and three-dimensional cases in a unified fashion.

Mathematics subject classification: 65N12, 65N15, 65N30, 65N50, 35J25.
Key words: Finite element method, Adaptive algorithm, Hanging node, 1-irregular mesh, Convergence analysis.

1. Introduction

The adaptive finite element method (AFEM) is an efficient and reliable tool in the numerical solution of partial differential equations. The typical structure of the adaptive algorithm is made up of four modules: "Solve", "Estimate", "Mark", and "Refine". Even though adaptivity has been a fundamental tool of engineering and scientific computing for about three decades, the convergence analysis is rather recent. It started with Döfler [15], who introduced a crucial marking (from now on called Döfler's marking) and proved the strict energy reduction for the Laplacian provided the initial mesh $\mathcal{T}_0$ satisfies a fineness assumption. By introducing the concept of data oscillation and the interior node property, Morin *et al.* [21, 22] removed restriction on the initial mesh $\mathcal{T}_0$ and proved the convergence of AFEM. Very recently, Cascon *et al.* established the convergence of the self-adjoint second order elliptic problem without interior node property [9]. All of these results are based on an important tool, i.e., Galerkin-orthogonality. There are some results about nonstandard finite element methods in the literature. Carstensen and Hoppe proved the convergence of adaptive nonconforming and mixed finite element methods [7, 8]. One key ingredient of these papers is the so-called "quasi-orthogonality". This

* Received December 31, 2008 / Revised version received July 30, 2009 / Accepted November 6, 2009 / Published online May 1, 2010 /

technique is extended to the high order mixed finite element methods for the Poisson equation in [11]. So far, all the theoretical results have been limited to triangular or tetrahedral meshes.

The objective of this paper is to study the convergence of the adaptive conforming quadrilateral and hexahedral element methods. Since quadrilateral and hexahedral elements have been widely used in practical computing, it is important to study the adaptive algorithms for these general non-affine equivalent finite elements. As we know, local refinements on triangular or tetrahedral meshes are well developed, including newest-vertex-bisection, longest edge bisection and red-green refinement. However, the implementation of local refinement on quadrilateral and hexahedral meshes is, in some sense, more difficult than that on triangular or tetrahedral meshes. Nowadays, most researchers in the field of adaptive quadrilateral or hexahedral element methods use the so called 1-irregular mesh (see Section 3). By establishing some lemmas such as quasi-orthogonality, estimator reduction and so on, we finally prove the convergence of adaptive finite element methods on 1-irregular quadrilateral and hexahedral meshes for the general second-order elliptic partial equations, in which we can conquer the difficulties due to the non-affine mapping.

The rest of this paper is organized as follows. In the next section, we present the preliminary including the notation and the problem under consideration which is followed by the description of some concepts like shape regularity, hanging node and 1-irregular mesh. In Section 4, we prove the convergence of the corresponding adaptive algorithms. Since 1-irregular meshes are not conforming, the degrees of freedom on edges with hanging nodes must be constrained. The problem of how to assemble a symmetric positive definite stiff matrix will be discussed in Section 5 which also covers some numerical experiments. Conclusions will be presented in Section 6.

2. Problem and General Notations

Let $\Omega \in \mathbb{R}^d$ ($d \in \{2,3\}$) be a bounded, polyhedral domain with boundary $\Gamma := \partial\Omega$. We assume that the initial mesh $\mathcal{T}_0$ is a conforming quadrilateral or hexahedral partition of the domain Ω. We consider a homogeneous Dirichlet boundary value problem for a linear second order elliptic partial differential equation(PDE):

$$\begin{cases} \mathcal{L}u := -\mathrm{div}(\mathbf{A}\nabla u) + \mathbf{b}\cdot\nabla u + cu = f, & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

We assume

- $\mathbf{A} = (a_{ij})_{d\times d}$: $\Omega \mapsto \mathbb{R}^{d\times d}$ is symmetric positive definite and V-elliptic on Ω and $a_{ij} \in W^{1,\infty}(\Omega)(i,j = 1,2,\cdots,d)$;
- $\mathbf{b} = (b_k)_{d\times 1} \in (W^{1,\infty}(\Omega))^d$; $\quad c \in L^\infty(\Omega)$ and $c \geq 0$; $\quad f \in L^2(\Omega)$.

The weak formulation of (2.1) reads as follows: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := (\mathbf{A}\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u, v) + (cu, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{2.2}$$

We denote by $\|\cdot\|_{a,\Omega}$ the energy norm

$$\|w\|_{a,\Omega}^2 := \int_\Omega \mathbf{A}\nabla w\cdot\nabla w + cw^2, \quad \forall w \in H_0^1(\Omega),$$