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SUPER-GEOMETRIC CONVERGENCE OF A SPECTRAL ELEMENT METHOD FOR EIGENVALUE PROBLEMS WITH JUMP COEFFICIENTS*

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Abstract

We propose and analyze a C^0 spectral element method for a model eigenvalue problem with discontinuous coefficients in the one dimensional setting. A super-geometric rate of convergence is proved for the piecewise constant coefficients case and verified by numerical tests. Furthermore, the asymptotical equivalence between a Gauss-Lobatto collocation method and a spectral Galerkin method is established for a simplified model.

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1. Introduction

We often encounter eigenvalue problems with discontinuous coefficients in practice. Examples of such applications may be found in [11]. In this paper, we consider the following one dimensional model problem: Find $(\lambda, u) \in \mathbb{R}^+ \times H^2(-\pi,\pi)$ such that

$$-u''(x) = \lambda c(x)u(x), \qquad u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi). \tag{1.1}$$

Here $c(x) \geq c_0 > 0$ is a piecewise constant, or piecewise analytic function. The physics background of this model problem comes from the source-free Maxwell equations describing the transverse-magnetic mode in the one-dimensional periodic media, where the function u represents the electric field pattern, and the dielectric function $c(x)$ describes a unit cell from a multilayer structure with 2π-periodicity. This model problem was discussed by Min and

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Gottlieb in [11] where C^1 conforming spectral collocation methods were constructed on two elements over

$$H^2_{per}(-\pi,\pi) = \{v \in H^2(-\pi,\pi):\ \ v(-\pi) = v(\pi),\ v'(-\pi) = v'(\pi)\},$$

and error bounds of type $\mathcal{O}(p^{-m})$ were established. Note that the solution of (1.1) belongs to C^1.

It would be interesting to discuss C^0 spectral element methods over

$$H^1_{per}(-\pi,\pi) = \{v \in H^1(-\pi,\pi):\ \ v(-\pi) = v(\pi)\},$$

since the construction of a C^0 spectral element method is much simpler than that of the global C^1 spectral collocation method proposed in [11]. The idea of the spectral element can be found, e.g., in an early work [12]. Note that the spectral element method is equivalent to the so-called p-version finite element method, see e.g., [3]. Under the finite element variational framework, we are able to prove a super-geometric error bound of type $\mathcal{O}(e^{-2p(\log p-\gamma)})$. In some earlier works of the third author, the super-geometric error bound of type $\mathcal{O}(e^{-p(\log p-\gamma)})$ has been established for some spectral/collocation approximations of the two-point boundary problem [17, 18]. Our error bound for the eigenvalue approximation "doubles" the error bound for the associated eigenfunction approximation, the fact we have known for the h-version finite element method. It is worthy to point out that in the literature of the spectral method, it is a common practice to consider error bounds of type $\mathcal{O}(p^{-m})$, see, e.g., [5–7, 10, 15, 16], and reference therein. To the best of our knowledge, this is the first time that a super-geometric convergence rate is established for the eigenvalue approximation by the spectral method.

2. Theoretical Setting

The variational formulation of (1.1) is to find $(\lambda, u) \in \mathbb{R}^+ \times H^1_{per}(-\pi,\pi)$ such that

$$(u', v') = \lambda(cu, v), \quad \forall v \in H^1_{per}(-\pi,\pi). \tag{2.1}$$

In this paper, we also consider the Dirichlet problem

$$-u''(x) = \lambda c(x) u(x), \qquad u(0) = 0 = u(1).$$

Its variational formulation is to find $(\lambda, u) \in \mathbb{R}^+ \times H^1_0(0,1)$ such that

$$(u', v') = \lambda(cu, v), \qquad \forall v \in H^1_0(0,1). \tag{2.2}$$

By the general theory [2, 8], both problems (2.1) and (2.2) have countable infinite sequence of eigen-pairs (λ_j, u_j) satisfying

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty, \qquad (u_i', u_j') = \lambda_j(cu_i, u_j) = \lambda_j \delta_{ij}.$$

Furthermore, eigenvalues can be characterized as extrema of the Rayleigh quotient $R(u) = (u', u')/(cu, u)$ as follows

$$\lambda_1 = \inf_{u \in S} = R(u_1),$$

$$\lambda_k = \inf_{u\in S,\ (u', u_j')=0, j=1,\dots,k-1} R(u) = R(u_k), \quad k = 2, 3, \dots,$$