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SUPERCONVERGENCE OF GRADIENT RECOVERY SCHEMES ON GRADED MESHES FOR CORNER SINGULARITIES*

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Abstract

For the linear finite element solution to the Poisson equation, we show that superconvergence exists for a type of graded meshes for corner singularities in polygonal domains. In particular, we prove that the L^2-projection from the piecewise constant field ∇u_N to the continuous and piecewise linear finite element space gives a better approximation of ∇u in the H^1-norm. In contrast to the existing superconvergence results, we do not assume high regularity of the exact solution.

Mathematics subject classification: 65N12, 65N30, 65N50.
Key words: Superconvergence, Graded meshes, Weighted Sobolev spaces, Singular solutions, The finite element method, Gradient recovery schemes.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We shall consider the linear finite element approximation for the Poisson equation

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

We are interested in the case when Ω is concave, and thus the solution of (1.1) possesses corner singularities at vertices of Ω where some of the interior angles are greater than π.

By the regularity theory, the solution u is in $H^{1+\beta}(\Omega)$ with $\beta = \min_i\{\pi/\alpha_i, 1\}$, where α_i are interior angles of the polygonal domain Ω. It is easy to see that when the maximum angle is larger than π, i.e., Ω is concave, $u \notin H^2(\Omega)$, and thus the finite element approximation based on quasi-uniform grids will not produce the optimal convergence rate. Graded meshes near the singular vertices are employed to recovery the optimal convergence rate. Such meshes can be constructed based on a priori estimates [3,4,6,24,25,31,37] or on a posteriori analysis [9,12,39]. In this paper, we shall consider the approach used in [6,31], and in particular, focus on the linear finite element approximation of (1.1).

In [6,31], a sequence of linear finite element spaces $\mathbb{V}_N \subset H_0^1(\Omega)$ is constructed, such that

$$\|\nabla(u - u_N)\|_{L^2(\Omega)} \le CN^{-1/2}\|f\|_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega), \tag{1.2}$$

where $u_N = u_{\mathbb{V}_N}$ is the finite element approximation and $N = \dim \mathbb{V}_N$. The convergence rate $N^{-1/2}$ in (1.2) is the best possible rate we can expect for the linear element, and the solution

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u_N is the best approximation (i.e., the projection) of u into $\mathbb{V}_N$ in the H^1 semi-norm. We cannot find a better approximation to u in the space $\mathbb{V}_N$ measured in the H^1 semi-norm.

The main contribution of this paper is to demonstrate that appropriate post-processing of the piecewise constant vector function ∇u_N will improve the convergence rate. More precisely, let $\overline{\mathbb{V}}_N$ denote the space of continuous and piecewise linear finite element functions. Note that $\overline{\mathbb{V}}_N$ is bigger than $\mathbb{V}_N$ since it also contains nodal basis of boundary nodes. For any $u \in L^2(\Omega)$, denote by

$$Q_N : L^2(\Omega) \mapsto \overline{\mathbb{V}}_N, \quad (Q_N u, v_n)_{L^2} := (u, v_n)_{L^2}, \quad \forall v_n \in \overline{\mathbb{V}}_N,$$

the L^2-projection to $\overline{\mathbb{V}}_N$, and for $u \in H^1(\Omega)$,

$$Q_N(\nabla u) := Q_N(\partial_x u, \partial_y u) = (Q_N(\partial_x u), Q_N(\partial_y u)) \in \overline{\mathbb{V}}_N \times \overline{\mathbb{V}}_N.$$

Then on appropriate graded meshes and for any $\delta > 0$, we shall show

$$\|\nabla u - Q_N(\nabla u_N)\|_{L^2(\Omega)} \le C N^{-5/8+\delta} \|f\|_{H^1(\Omega)}, \quad \forall f \in H^1(\Omega), \tag{1.3}$$

where C depends only on the interior angles of Ω, the initial triangulation $\mathcal{T}_0$ of Ω, and the constant δ. Therefore, we obtain a better approximation of ∇u based on existing information on the mesh and corresponding matrices. Note that instead of the inversion of the stiffness matrix, the computation of $Q_N(\nabla u_N)$ only involves the inversion of the mass matrix. Following our diagonal scaling technique in Section 2, the preconditioned conjugate gradient (PCG) method with the diagonal pre-conditioner will be convergent very quickly. Consequently, the computational cost of $Q_N u_N$ is negligible comparing with that of u_N.

The improved convergence rate (1.3) is known as superconvergence in the literature. Let $u_I \in \mathbb{V}_N$ be the nodal interpolation of u. Our proof of (1.3) is based on the following super-closeness between u_N and u_I in $\mathbb{V}_N$:

$$\|\nabla u_I - \nabla u_N\|_{L^2(\Omega)} \le C N^{-5/8+\delta} \|f\|_{H^1(\Omega)}, \quad \forall f \in H^1(\Omega). \tag{1.4}$$

Our approach can be easily modified to prove a similar result for average type recovery scheme [47] or polynomial preserving recovery scheme [45]. For example, let us define an average type recovery scheme by $R : \nabla \mathbb{V}_N \mapsto \overline{\mathbb{V}}_N \times \overline{\mathbb{V}}_N$

$$R(\nabla u_N)(x_i) = \frac{\sum_{\tau \in \omega_i} |\tau| \nabla u_N|_\tau}{|\omega_i|}, \text{ for all vertices } x_i \in \mathcal{T},$$

where ω_i is the patch including the vertex x_i, i.e., the union of all triangles containing x_i, and $|\cdot|$ is the two dimensional Lebesgue measure. Then a similar estimate

$$\|\nabla u - R(\nabla u_N)\|_{L^2(\Omega)} \le C N^{-5/8+\delta} \|f\|_{H^1(\Omega)}, \quad \forall f \in H^1(\Omega), \tag{1.5}$$

holds. The average type recovery involves only simple function evaluation and arithmetic operations, and thus is more computationally favorable.

The idea of post-processing the solution in the L^2-norm for a better approximation has been widely addressed. For example, see the early paper [21] in 1974. When the solution u is smooth enough, the superconvergence theory is well estabilished. See [5, 7, 10, 13–15, 27, 29, 36, 38, 46] for the super-closness (1.4); see [7, 15, 22, 28, 30, 41–44] for the superconvergence of recovered gradient (1.3) or (1.5). Analogue of (1.3), (1.4), and (1.5) on quasi-uniform meshes are usually proved with the assumption $u \in H^3(\Omega) \cap W^{2,\infty}(\Omega)$, which is not realistic for corner singularities.