Journal of Computational Mathematics
Vol.27, No.4, 2009, 543–560. http://www.global-sci.org/jcm
doi:10.4208/jcm.2009.27.4.018

RECOVERY A POSTERIORI ERROR ESTIMATES FOR GENERAL CONVEX ELLIPTIC OPTIMAL CONTROL PROBLEMS SUBJECT TO POINTWISE CONTROL CONSTRAINTS*

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Abstract

Superconvergence and recovery a posteriori error estimates of the finite element approximation for general convex optimal control problems are investigated in this paper. We obtain the superconvergence properties of finite element solutions, and by using the superconvergence results we get recovery a posteriori error estimates which are asymptotically exact under some regularity conditions. Some numerical examples are provided to verify the theoretical results.

Mathematics subject classification: 49J20, 65N30.
Key words: General convex optimal control problems, Finite element approximation, Control constraints, Superconvergence, Recovery operator.

1. Introduction

Efficient numerical methods are essential to successful applications of optimal control problems (see, e.g., [17, 27, 33]) in practical areas. It is well known that finite element methods are undoubtedly the most widely used numerical methods in solving optimal control problems. There have been extensive studies in convergence of the finite element approximation for various optimal control problems (see, e.g., [2, 14, 16, 21, 22, 34]). Recently, a priori error estimates of the finite element approximation for optimal control problems governed by linear state equations can be found in [3], and a posteriori error estimates in [4, 5, 19, 20, 25, 28–31]. Some primary works on sharp a posteriori error estimates and a priori error estimates of mixed

* Received December 21, 2007 / Revised version received November 15, 2008 / Accepted December 25, 2008 /

finite element methods for optimal control problems were obtained in [11–13, 36, 37]. Adaptive finite element methods are among the most important classes of numerical methods to boost accuracy and efficiency of the finite element discretization. The literature in this area is huge, see, e.g., [1, 6, 35, 40, 42–44]. The superconvergence property of finite element solutions has also been an active research area in numerical analysis for optimal control problems (see, e.g., [10, 26, 32, 39]). Very recently, superconvergence of mixed finite element methods for optimal control problems has been studied in [7–9, 12, 41]. The main objective for investigating the superconvergence property is to improve the existing approximation accuracy by applying certain postprocessing techniques which are easy to implement. For the quadratic optimal control problems, some superconvergence results have been established (see [26,39]). In [18], Hinze presented a method that is not necessary to discretize the control variable for linear quadratic optimal control problems.

Finite element recovery techniques are post-processing methods that reconstruct numerical approximations from finite element solutions to achieve better results. To be practically useful, a good recovery method should have the following three features: (i) It is simple to implement and cost effective. In practice, a recovery procedure takes only very small portion of the whole computation cost; (ii) It is applicable to higher dimensions; and (iii) It is problem independent, i.e., a recovery process uses only numerical solution data.

This paper is concerned with the following general convex optimal control problem:

$$\min_{u\in K\subset L^2(\Omega_U)} \{g(y)+h(u)\} \tag{1.1}$$

$$-\operatorname{div}(A\nabla y)+a_0y=f+Bu, \qquad \text{in } \Omega\ , \tag{1.2}$$

$$y=0, \qquad \text{on } \partial\Omega, \tag{1.3}$$

where g and h are convex functionals, K is a closed convex set in $L^2(\Omega_U)$, Ω and Ω_U are two bounded open subsets in R^n $(n\le 3)$ with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$, respectively. Let f be a given function of the space $L^2(\Omega)$ and B be a continuous linear operator from $L^2(\Omega_U)$ to $L^2(\Omega)$. The coefficient matrix $A(\cdot)=(a_{ij}(\cdot))_{n\times n}$ is symmetric and positive definite. Moreover, we require $0\leqslant a_0\in L^\infty(\Omega)$.

Denote by $W^{m,p}(\Omega)$ the usual Sobolev space on Ω with norm and semi-norm defined by

$$\begin{aligned}\|\phi\|^p_{m,p,\Omega} &= \sum_{|\alpha|\le m}\int_\Omega |\ \partial^\alpha\phi\ |^p\ dx,\\ |\ \phi\ |^p_{m,p,\Omega} &= \sum_{|\alpha|=m}\int_\Omega |\ \partial^\alpha\phi\ |^p\ dx,\end{aligned}$$

where $\phi\in W^{m,p}(\Omega)$. We set $W_0^{m,p}(\Omega)=\{\phi\in W^{m,p}(\Omega):\phi\ |_{\partial\Omega}=0\}$. In particular, we write $H^m(\Omega)=W^{m,2}(\Omega)$ $(H_0^m(\Omega)=W_0^{m,2}(\Omega))$ and $\|\cdot\|_{m,\Omega}=\|\cdot\|_{m,p,\Omega}$ $(\|\cdot\|_{H_0^m(\Omega)}=\|\cdot\|_{W_0^{m,p}(\Omega)})$, $|\cdot|_{m,\Omega}=|\cdot|_{m,p,\Omega}$ $(|\cdot|_{H_0^m(\Omega)}=|\cdot|_{W_0^{m,p}(\Omega)})$ for $p=2$. Besides, c or C denotes a general positive constant independent of h.

In this paper, we adopt the same recovery operators mentioned in [26] to solve general convex optimal control problems. We get the superconvergence property of finite element solutions, by which recovery a posteriori error estimates are obtained. The control variable is approximated by piecewise constant functions, and both the state y and the co-state p by piecewise linear finite element functions. We prove the superconvergence error estimate in L^2-norm between the approximated solution and the L^2-projection of the control, and superconvergence error