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CONVERGENCE OF AN IMMERSED INTERFACE UPWIND SCHEME FOR LINEAR ADVECTION EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS II: SOME RELATED BINOMIAL COEFFICIENT INEQUALITIES *

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Abstract

In this paper we give proof of three binomial coefficient inequalities. These inequalities are key ingredients in [Wen and Jin, J. Comput. Math. 26, (2008), 1-22] to establish the L^1-error estimates for the upwind difference scheme to the linear advection equations with a piecewise constant wave speed and a general interface condition, which were further used to establish the L^1-error estimates for a Hamiltonian-preserving scheme developed in [Jin and Wen, Commun. Math. Sci. 3, (2005), 285-315] to the Liouville equation with piecewise constant potentials [Wen and Jin, SIAM J. Numer. Anal. 46, (2008), 2688-2714].

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1. Introduction

In this paper we give proof of the following three binomial coefficient inequalities.

Theorem 1.1.

$$\sum_{l=0}^{n} \Gamma_{n,l}(\lambda)|n-n\lambda-l| \le \sqrt{\frac{2}{e}}\sqrt{\lambda(1-\lambda)(n+1)}, \quad \forall 0<\lambda<1, n\in\mathbb{N}, \tag{1.1}$$

where

$$\Gamma_{n,l}(\lambda) = C_n^l \lambda^{n-l}(1-\lambda)^l, \tag{1.2}$$

and C_n^l denote binomial coefficients.

Theorem 1.2. *Let $0<\lambda^-,\lambda^+<1, n\in\mathbb{N}, J\in\mathbb{Z},$ with $-n\lambda^-<J<0, K=\frac{\lambda^+}{\lambda^-}(J+n\lambda^-)$. Define*

$$T_1 = \nu(n,n+J+1,\lambda^-)\lambda^-, \quad \text{if } [K]^- = 0, \tag{1.3}$$

$$\begin{aligned} T_1 = {} & \nu(n,n+J+1,\lambda^-)\lambda^- + \sum_{l=n+J+1-[K]^-}^{n-1} \sum_{j=\max(n+1-l-[K]^-,1)}^{\min(n-l,-J)} \sum_{k=0}^{l} \Lambda_{j,k,l}^n \\ & + \nu(n,n+1-[K]^-,\lambda^+)\lambda^+, \quad \text{if } [K]^- > 0, \end{aligned} \tag{1.4}$$

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where $[x]^-$ *denotes the largest integer no more than* x, *and*

$$\nu(n,p,z)=\sum_{l=p}^{n}\Gamma_{n,l}(z)z^{-1}(l-p+1),\quad 0\le p\le n, 0<z<1, \tag{1.5}$$

$$\Lambda^n_{i,j,k}=C^k_{j+k-1}C^{l-k}_{n-j-k}(\lambda^+)^{n-l-j+1}(1-\lambda^+)^{l-k}(\lambda^-)^{j-1}(1-\lambda^-)^k. \tag{1.6}$$

Then

$$T_1\le\nu(n,n-[n\lambda^m]^++1,\lambda^m)\lambda^M, \tag{1.7}$$

where $[x]^+$ *denotes the smallest integer no less than* x, $\lambda^m=\min\{\lambda^-,\lambda^+\}$, *and* $\lambda^M=\max\{\lambda^-,\lambda^+\}$.

Theorem 1.3. *Let* $0<\lambda^-,\lambda^+<1, n\in\mathbb{N}, J\in\mathbb{Z}$, *with* $-n\lambda^-<J<0, K=\frac{\lambda^+}{\lambda^-}(J+n\lambda^-)$. *Define*

$$T_2=\sum_{l=0}^{n+J-[K]^+-1}\ \sum_{j=1-J}^{n-l-[K]^+}\ \sum_{k=0}^{l}\Lambda^n_{i,j,k}. \tag{1.8}$$

Then

$$T_2\le\eta(n,n-[n\lambda^m]^--1,\lambda^m)\lambda^M, \tag{1.9}$$

where

$$\eta(n,p,z)=\sum_{l=0}^{p}\Gamma^l_n(z)z^{-1}(p+1-l),\quad 0\le p\le n-1, 0<z<1. \tag{1.10}$$

These binomial coefficient inequalities have been used in [6] to derive the L^1-error estimates for the upwind difference scheme to the linear advection equation

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0,\qquad t>0, x\in\mathbb{R}, \tag{1.11}$$

$$u|_{t=0}=u_0(x), \tag{1.12}$$

with a step function wave speed

$$c(x)=\begin{cases} c^- & x<0,\\ c^+ & x>0,\end{cases} \tag{1.13}$$

where we consider $c(x)$ has definite sign.

Eqs. (1.11)-(1.13) is the simplest case of a hyperbolic equation with singular (discontinuous or measure-valued) coefficients. In [6], we proved that given a general interface condition

$$u(0^+,t)=\rho u(0^-,t),\qquad \rho>0, \tag{1.14}$$

the upwind difference scheme with the immersed interface condition converges in L^1-norm to Eqs. (1.11)-(1.13) with the corresponding interface condition, and derived the half-order L^1-error bounds with explicit coefficients for the numerical solutions. Due to the linearities of both Eq. (1.11) and the upwind difference scheme, the error estimates for general BV initial data can be derived based on error estimates for some Riemann initial data. This strategy is specifically suitable for linear schemes and linear equations and has been used in [5] to estimate lower error