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STABLE FOURTH-ORDER STREAM-FUNCTION METHODS FOR INCOMPRESSIBLE FLOWS WITH BOUNDARIES*

Thomas Y. Hou
Department of Applied and Computational Mathematics, California Institute of Technology, Pasadena, CA, 91125 USA
Email: hou@acm.caltech.edu
Brian R. Wetton
Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2, Canada
Email: wetton@math.ubc.ca

Abstract

Fourth-order stream-function methods are proposed for the time dependent, incompressible Navier-Stokes and Boussinesq equations. Wide difference stencils are used instead of compact ones and the boundary terms are handled by extrapolating the stream-function values inside the computational domain to grid points outside, up to fourth-order in the no-slip condition. Formal error analysis is done for a simple model problem, showing that this extrapolation introduces numerical boundary layers at fifth-order in the stream-function. The fourth-order convergence in velocity of the proposed method for the full problem is shown numerically.

Mathematics subject classification: 65M12, 76D05.
Key words: Incompressible flow, Stream-function formulation, Finite difference methods.

1. Introduction

Numerical methods based on the vorticity stream-function formulation of the Navier-Stokes equations have been used with success for flows in two-dimensions (2D). There are many references to second-order finite difference methods applied to different problems: the driven cavity problem [20]; flow past a cylinder [16]; and flow in tubes with occlusions [14] for example. Considerable effort has been spent in developing higher order finite difference methods for the vorticity stream-function formulation, including the early work in [1, 7, 10, 11]. These authors use some combination of compact differencing and one-sided differencing near the boundary to develop fourth-order methods. More modern approaches [3, 9] use the stream-function directly, without reference to vorticity. These methods also use compact differencing. In this paper, we develop a new fourth-order method for the time dependent Navier-Stokes and Boussinesq Equations that uses a wide stencil rather than compact differencing. This approach offers additional flexibility in choice of time stepping and applicability in mapped grids over other methods. Formal analysis of a simple model problem indicates that the method will give fourth-order accuracy in computed velocities. This is demonstrated for the full problem in numerical convergence studies.

There are several reasons to consider higher order methods. Of course, the underlying reason is that we wish to get an approximation to a flow to a given accuracy or resolution using less computational time and less storage space. This can manifest itself in many ways. In some cases, high accurate benchmark solutions are required [11]. In other cases, the higher

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order accuracy is required to resolve delicate phenomena [1, 16]. The results of Kreiss [17] and Chang [6] show that fourth-order methods are in some sense "optimal". Their results show that fourth-order methods outperform second-order methods for 2D model convection-diffusion problems. They also show that if only modest accuracy (1%) is required that fourth-order methods are at least as efficient as higher order finite difference methods and more efficient than second-order and pseudo-spectral methods. Of course, if very high accurate solutions are needed, pseudo-spectral methods are the best to use.

There are two basic approaches to the construction of higher order finite difference methods for the time-dependent Navier-Stokes equations and more generally, any convection diffusion problem with smooth solutions. One approach is the use of compact differencing which gets its name from the fact that the resulting method is based on a difference stencil no larger than that needed for a second-order method. There are many variants of this approach including [7,10,11]. The method in [1] combines compact differencing for the stream-function equation with an integral constraint technique to solve for the boundary vorticity. Strictly speaking, what we will call compact differencing from now on in this paper is actually a restricted subset known as Operator Compact Implicit (OCI) differencing in which only the physical unknowns are used on grid points (see [4] for a good discussion). In another type of compact differencing, called simply Compact Implicit (CI), derivatives of the unknowns are introduced at each grid point. This is the approach taken in [3]. The introduction of the extra unknowns effectively widens the difference stencil and extrapolation or one sided differencing must be used to eliminate the derivative values on the boundary (see [11]) in higher order methods, so CI are similar in spirit to the wide methods discussed below. We will not pursue CI methods in this paper.

The approach taken in this work is to use standard fourth-order differencing using a wider difference stencil than needed for second-order accuracy. We call these wide methods to distinguish them from compact methods. For periodic problems this approach is very natural, but for problems with boundaries, these methods require values on more grid points outside the boundary than can be eliminated using the boundary conditions of the problem. The authors propose below a method of "extrapolating" the values outside the grid using values inside the grid based on their earlier work in vorticity boundary conditions for second-order schemes [12]. The values at grid points outside the boundary are related to the values inside the boundary in such a way that they satisfy the boundary conditions to fourth-order accuracy. For stream-function methods, it is the no-slip condition that is matched to fourth-order to give velocities that are fourth-order accurate.

In the next section, the 2D Navier-Stokes and Boussinesq approximations are introduced. Then, wide and compact schemes are introduced and applied to a 1D model problem. Error analysis of the wide scheme for the model problem is conducted, using matched interior and numerical boundary layer errors as presented in [22]. This is followed by a section on the specification of fourth-order wide methods for the 2D incompressible fluid flow equations. These methods are then applied to two 2D example problems in fluid flow where the fourth-order convergence in velocities (and temperature for the Boussinesq flow example) of the method is verified.

2. The Navier-Stokes and Boussinesq Equations

The 2D Navier-Stokes equations in primitive variables (velocity $\mathbf{u} = (u, v)$ and pressure p) are