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THE HP-VERSION OF BEM – FAST CONVERGENCE, ADAPTIVITY AND EFFICIENT PRECONDITIONING*

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Abstract

In this survey paper we report on recent developments of the *hp*-version of the boundary element method (BEM). As model problems we consider weakly singular and hypersingular integral equations of the first kind on a planar, open surface. We show that the Galerkin solutions (computed with the *hp*-version on geometric meshes) converge exponentially fast towards the exact solutions of the integral equations. An *hp*-adaptive algorithm is given and the implementation of the *hp*-version BEM is discussed together with the choice of efficient preconditioners for the ill-conditioned boundary element stiffness matrices. We also comment on the use of the *hp*-version BEM for solving Signorini contact problems in linear elasticity where the contact conditions are enforced only on the discrete set of Gauss-Lobatto points. Numerical results are presented which underline the theoretical results.

Mathematics subject classification: 65N55.
Key words: *hp*-version of the boundary element method, Adaptive refinement, Preconditioning, Signorini contact.

1. Exponential Convergence

In this paper we consider the *hp*-version of the boundary element method (BEM) for Dirichlet and Neumann screen problems of the Laplacian in $\mathbb{R}^3\backslash\overline{\Gamma}$, where Γ is a planar surface piece with polygonal boundary (for details see also the survey paper [18]). That is, given f or g on Γ find $u \in \mathbb{R}^3\backslash\overline{\Gamma}$ satisfying

$$
\begin{aligned}
&\Delta u = 0 \text{ in } \mathbb{R}^3\backslash\bar{\Gamma},\\
&u = f \in H^{1/2}(\Gamma) \text{ (Dirichlet)} \quad \text{or} \quad \frac{\partial u}{\partial n} = g \in H^{-1/2}(\Gamma)\text{(Neumann)},\\
&u = \mathcal{O}(|x|^{-1}) \quad \text{as } |x| \to \infty.
\end{aligned}
$$

These exterior boundary value problems are called screen problems and can be formulated equivalently as first kind integral equations with weakly singular and hypersingular kernels, namely

$$V\psi(x) := \frac{1}{2\pi}\int_\Gamma \frac{1}{|x-y|}\psi(y)\,ds_y = 2f(x),\ \ x \in \Gamma \text{ (Dirichlet)}, \tag{1.1}$$

$$Wv(x) := -\frac{1}{2\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\frac{1}{|x-y|}v(y)\,ds_y = 2g(x),\ \ x \in \Gamma \text{ (Neumann)}. \tag{1.2}$$

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As we have shown in [17] these integral equations have unique solutions $\psi \in \tilde{H}^{-1/2}(\Gamma)$, $v \in \tilde{H}^{1/2}(\Gamma) = H_{00}^{1/2}(\Gamma)$.

The Galerkin boundary element schemes for (1.1) and (1.2) read with the L^2-duality on Γ $\langle \cdot, \cdot \rangle$: Find $\psi_N \in S^0_{h,p}$

$$\langle V\psi_N, \phi_N \rangle = \langle 2f, \phi_N \rangle, \quad \forall \phi_N \in S^0_{h,p} \subset \tilde{H}^{-1/2}(\Gamma), \tag{1.3}$$

and find $v_N \in S^1_{h,p}$

$$\langle W v_N, w_N \rangle = \langle 2g, w_N \rangle, \quad \forall w_N \in S^1_{h,p} \subset \tilde{H}^{1/2}(\Gamma). \tag{1.4}$$

Since the operators V and W define coercive, continuous bilinear forms we immediately have quasi-optimality of the Galerkin errors:

$$\begin{aligned} \|\psi - \psi_N\|_{\tilde{H}^{-1/2}(\Gamma)} &\lesssim \operatorname{dist}\big(\psi, S^0_{h,p}(\Gamma)\big), \\ \|v - v_N\|_{\tilde{H}^{1/2}(\Gamma)} &\lesssim \operatorname{dist}\big(v, S^1_{h,p}(\Gamma)\big). \end{aligned}$$

For the screen problems above these estimates yield only very low order of convergence rate $\mathcal{O}(h^{1/2-\varepsilon} p^{-1+2\varepsilon})$ with arbitrary $\varepsilon > 0$ (see, e.g., [4, 15, 16]).

The indices h and p in the notation for the trial spaces $S^0_{h,p}(\Gamma)$ and $S^1_{h,p}(\Gamma)$ refer to h- and p-versions, respectively; where in the h-version a more accurate Galerkin solution is obtained by mesh refinement (and the polynomial degree p is kept fixed) whereas in the p-version a higher accuracy is obtained by increasing the polynomial degree on the same mesh. The implementation of the h-version is standard. In the p-version BEM for the weakly singular integral equation we use tensor products of Legendre polynomials on rectangular meshes and for the hypersingular integral equation we take instead antiderivatives of Legendre polynomials. On triangular meshes more sophisticated trial functions must be used, as we will show further below.

If one uses a geometric mesh refinement together with a properly chosen polynomial degree distribution one obtains even exponentially fast convergence rates for the Galerkin errors of the above integral equations. We have the following result proven in [1] for $d = 2$ and in [6, 9, 13] for $d = 3$ where d denotes the spatial dimension; i.e., Γ is polygon for $d = 2$, and Γ is a planar surface piece if $d = 3$.

Theorem 1.1. *For given piecewise analytic functions f, g in* (1.1) *and* (1.2) *and corresponding Galerkin solutions $\psi_N \in S^0_{h,p-1}(\Gamma^n_\sigma)$, $v_N \in S^1_{h,p}(\Gamma^n_\sigma)$ of* (1.3) *and* (1.4) *on the geometric mesh Γ^n_σ there holds*

$$\left.\begin{aligned} &\|\psi - \psi_N\|_{\tilde{H}^{-1/2}(\Gamma)} \\ &\|v - v_N\|_{\tilde{H}^{1/2}(\Gamma)} \end{aligned}\right\} \leq \begin{cases} C \exp(-b\sqrt{N}), & d = 2, \\ C \exp(-b\sqrt[4]{N}) + \mathcal{O}(N^{-\alpha}), & d = 3, \end{cases}$$

with constants C, $b > 0$ independent of the dimension N of the trial space and arbitrary $\alpha > 0$.

The local mesh at a right angle corner of Γ is given in Fig. 1.1. The proof of the theorem is based on analysing the error in countably normed spaces and is based on the following lemma in [13].

Lemma 1.1. *For $u \in B^2_\beta(Q)$, $0 < \beta < 1$, there exists a spline $u_N \in S^1_{h,p}(Q^n_\sigma)$ and constants C, $b > 0$ independent of N, but dependent on σ, μ, β such that*

$$\|u - u_N\|_{H^1(Q)} \leq C\, e^{-b\sqrt[4]{N}}, \tag{1.5}$$

with $p_1 = 1$, $p_k = \max\big(2, [\mu(k-1)] + 1\big)$ $(k > 1)$ for $\mu > 0$.