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STABILIZED FEM FOR CONVECTION-DIFFUSION PROBLEMS ON LAYER-ADAPTED MESHES*

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Abstract

The application of a standard Galerkin finite element method for convection-diffusion problems leads to oscillations in the discrete solution, therefore stabilization seems to be necessary. We discuss several recent stabilization methods, especially its combination with a Galerkin method on layer-adapted meshes. Supercloseness results obtained allow an improvement of the discrete solution using recovery techniques.

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1. Introduction

We consider the two-dimensional convection-diffusion problem

$$Lu := -\varepsilon\Delta u - b\cdot\nabla u + cu = f \quad \text{in } \Omega \tag{1.1a}$$

$$u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{1.1b}$$

where ε is a small positive parameter, b, c are smooth and $f \in L_2(\Omega)$. Assuming

$$c + \frac{1}{2}\operatorname{div} b \geq \alpha_0 > 0, \tag{1.2}$$

the given problem admits a unique solution $u \in H_0^1(\Omega)$.

Let us introduce the ε- weighted H^1norm by

$$\|v\|_\varepsilon := \varepsilon^{1/2}|v|_1 + \|v\|_0.$$

Then for the Galerkin finite element method with piecewise linear or bilinear elements one can prove (C denotes a generic constant that is independent of ε and of the mesh)

$$\|u - u_h\|_\varepsilon \leq Ch|u|_2 \tag{1.3}$$

on quite general triangulations. However, estimate (1.3) is of no worth: in general, $|u|_2$ tends to infinity for $\varepsilon \to 0$ due to the presence of layers. The very weak stability properties of standard Galerkin lead to wild nonphysical oscillations in the discrete solution.

Therefore, stabilized Galerkin methods should be used. For several methods of this type—we shall present a short survey in Section 2—one has stability in a norm $\|\cdot\|_S$ that is stronger than the ε-weighted H^1 norm and, consequently, at most mild oscillations appear in the discrete solution. Moreover, for linear or bilinear elements we have the error estimate

$$\|u - u_h\|_S \leq C(\varepsilon^{1/2} + h^{1/2})\, h|u|_2. \tag{1.4}$$

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In contrast to the Galerkin method stabilized versions allow for local versions of (1.4) which show that stabilized methods provide good approximations in subdomains that exclude layers [40].

If one wants to resolve layers, it is possible to use layer-adapted meshes. These meshes are constructed a priori based on precise information on the structure of the layer (see Section 3). In this paper we mainly discuss problem (1.1) in $\Omega = (0,1)^2$ assuming

$$b = (b_1, b_2) > (\beta_1, \beta_2) > 0 \quad \text{on} \quad \overline{\Omega}. \tag{1.5}$$

Then exponential boundary layers form on

$$\Gamma^+ := \{x \in \Gamma : -b^T n > 0\}.$$

We shall also comment on the important special case $b_1 > \beta_1 > 0$, $b_2 \equiv 0$, which is characterized by the presence of parabolic boundary layers on

$$\Gamma^0 := \{x \in \Gamma : b^T n = 0\}$$

which are also called characteristic layers.

In case of exponential layers one has for standard Galerkin method applied to (1.1) on the simplest layer adapted mesh, the S-mesh (see Section 3), with a number of mesh points proportional to N^2

$$\|u - u^N\|_\varepsilon \le C N^{-1} \ln N, \tag{1.6}$$

see, e.g., [13, 44].

The fine mesh in the layer region induces some stability problems; nonetheless the computed solution exhibits oscillations (see the numerical experiments in [30]). Moreover, the stiffness matrix of the discrete problem generated has eigenvalues with large imaginary parts. Consequently, standard iterative methods are not able to solve the discrete systems efficiently. For the discussion of suitable iterative solvers see [14], Chapter 4. Let us remark, however, that robustness results for iterative solvers in the case of nonsymmetric problems are rare.

Therefore some stabilization seems to be necessary even when the layer-adapted meshes are used. A comparison of (1.3) and (1.6) suggests that for a stabilized method on a S-mesh one has

$$\|u - u^N\|_S \le C\big(\varepsilon^{1/2} + (N^{-1} \ln N)^{1/2}\big) N^{-1} \ln N.$$

But this is impossible because the estimate

$$\varepsilon^{1/2} |u - u^I|_1 \le C N^{-1} \ln N$$

for the interpolation error is optimal. Consequently, to verify the improved properties of stabilized methods on layer-adapted meshes in comparison to standard Galerkin we consider estimates for $\|u^I - u^N\|$ instead of $\|u - u^N\|$. In Section 4 we shall survey results of this type which often turn out to be *supercloseness results*. Furthermore, in many cases supercloseness allows the application of a *recovery procedure*. This yields an approximation Ru^N of u that is better than the approximation u^N computed first. Such recovery techniques are widely used in finite element methods to all kinds of problems including singularly perturbed problems, but for singularly perturbed problems its theoretical justification is especially delicate. Superconvergence and recovery techniques appear in several books of Chinese authors from the 1980s and 1990s but unfortunately these books are available only in Chinese with the recent exception [27].