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COMBINATION OF GLOBAL AND LOCAL APPROXIMATION SCHEMES FOR HARMONIC MAPS INTO SPHERES*

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Abstract

It is well understood that a good way to discretize a pointwise length constraint in partial differential equations or variational problems is to impose it at the nodes of a triangulation that defines a lowest order finite element space. This article pursues this approach and discusses the iterative solution of the resulting discrete nonlinear system of equations for a simple model problem which defines harmonic maps into spheres. An iterative scheme that is globally convergent and energy decreasing is combined with a locally rapidly convergent approximation scheme. An explicit example proves that the local approach alone may lead to ill-posed problems; numerical experiments show that it may diverge or lead to highly irregular solutions with large energy if the starting value is not chosen carefully. The combination of the global and local method defines a reliable algorithm that performs very efficiently in practice and provides numerical approximations with low energy.

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1. Introduction

We consider the simplest example of a geometric partial differential equation, namely, we study the problem of minimizing the Dirichlet energy among vector fields that satisfy boundary conditions and a pointwise unit length constraint. More precisely, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a positive integer m, and $u_{\rm D} \in H^{1/2}(\partial\Omega;\mathbb{R}^m)$ such that $|u_{\rm D}| = 1$ almost everywhere (a.e.) on $\partial\Omega$, we aim at finding (local) minimizers of the functional

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx \tag{1.1}$$

among maps

$$u \in \mathcal{A}(u_{\rm D}) := \big\{v \in H^1(\Omega;\mathbb{R}^m) : \, |v| = 1 \text{ a.e. in } \Omega, \;\; v|_{\partial\Omega} = u_{\rm D}\big\}.$$

The existence of (global) minimizers follows from the direct method in the calculus of variations provided that $\mathcal{A}(u_{\rm D}) \neq \emptyset$. Sufficient for this is that $u_{\rm D}$ is Lipschitz continuous on $\partial\Omega$, see [15]. Here, we restrict our attention to stationary points of E in $\mathcal{A}(u_{\rm D})$. These are the weak solutions of the Euler-Lagrange equations

$$-\Delta u = |\nabla u|^2 u \quad \text{in } \Omega, \quad |u| = 1 \quad \text{a.e. in } \Omega, \quad u|_{\partial\Omega} = u_{\rm D} \tag{1.2}$$

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and are called *harmonic maps*. The nonlinear partial differential equation (1.2) occurs as a greatly simplified subproblem in ferromagnetics [14] and liquid crystal theory [15,21] and serves as a model problem for partial differential equations into manifolds. The main difficulties in the development of convergent numerical methods are the nonconvexity of the constraint $|u| = 1$ a.e. in Ω, limited regularity and nonuniqueness of solutions of (1.2), as well as restricted flexibility of standard finite element methods. These problems have successfully been addressed in [1, 2]; the globally convergent iterative algorithm proposed and analyzed in those works realizes an H^1 gradient flow of E and computes stationary points of E in lowest order finite element spaces which satisfy the unit-length constraint at the nodes of the underlying triangulation. Weak subconvergence to a harmonic map and an energy decreasing property of the iteration are guaranteed if the underlying triangulations are weakly acute. The fact that the algorithm decreases the energy in each step is important, since it is known that harmonic maps may be discontinuous everywhere, cf. [18], whereas energy minimizing (or weakly stationary) harmonic maps are smooth away from a discrete set, cf. [10, 19, 20]; if $d = 2$ then harmonic maps are smooth [11] but may still fail to be unique. Although the algorithm of [1, 2] is capable to deal with related difficulties, it suffers from extremely slow convergence. The presumably more efficient solution of the discrete formulation by means of a Newton iteration is critical for various reasons. First, the iteration matrix may become singular and second, by nonconvexity of the constraint, the iteration may fail to converge even if it is well-posed. Nevertheless, when a good initial value is available then the Newton iteration typically converges rapidly to a solution of the nonlinear system of equations. In order to benefit from the best properties of the global and the local scheme, we propose to alternatingly perform a few iterations of each scheme. This leads to a reliable iteration that converges faster than the global strategy in all of our numerical experiments. For other approaches to the computation of harmonic maps into spheres we refer the reader to [15, 16]; for approximation results of harmonic maps into more general targets we refer to [3, 17].

The outline of this article is as follows. Preliminaries and notation are introduced in Section 2. The discrete scheme and its properties are discussed in Section 3. In Section 4 we recall the global solver of [1, 2] and define the local solution strategy which is based on a saddle point formulation with a separately convex augmented Lagrangian. The main contribution of this work is the combination of the global and local strategy and is stated in Section 5. Numerical experiments are reported in Section 6 and show that the global strategy is slowly convergent, that the local strategy may fail to converge at all, and that the combined strategy performs most efficiently in our examples.

2. Preliminaries

Throughout this paper we assume that $\mathcal{T}_h$ is a regular triangulation of the polygonal or polyhedral bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ into triangles or tetrahedra of maximal diameter h for $d = 2, 3$, respectively. The subscript h refers to the maximal mesh-size of $\mathcal{T}_h$, i.e., $h = \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$. When dealing with a sequence of triangulations, we assume that h belongs to a countable set of positive real numbers that accumulate at zero. We let $\mathcal{S}^1(\mathcal{T}_h) \subseteq H^1(\Omega)$ denote the lowest order finite element space on $\mathcal{T}_h$, i.e., $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)$ if and only if $\phi_h \in C(\overline{\Omega})$ and $\phi_h|_T$ is affine for each $T \in \mathcal{T}_h$. The subset $\mathcal{S}^1_0(\mathcal{T}_h) \subset \mathcal{S}^1(\mathcal{T}_h)$ consists of all functions in $\mathcal{S}^1(\mathcal{T}_h)$ that vanish on $\partial\Omega$, i.e., $\mathcal{S}^1_0(\mathcal{T}_h) := \mathcal{S}^1(\mathcal{T}_h) \cap H^1_0(\Omega)$. Given the set of all nodes (or vertices) $\mathcal{N}_h$ in $\mathcal{T}_h$ and letting $\big(\varphi_z : z \in \mathcal{N}_h\big)$ denote the nodal basis in $\mathcal{S}^1(\mathcal{T}_h)$, we define the