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A VARIATIONAL EXPECTATION-MAXIMIZATION METHOD FOR THE INVERSE BLACK BODY RADIATION PROBLEM*

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Abstract

The inverse black body radiation problem, which is to reconstruct the area temperature distribution from the measurement of power spectrum distribution, is a well-known ill-posed problem. In this paper, a variational expectation-maximization (EM) method is developed and its convergence is studied. Numerical experiments demonstrate that the variational EM method is more efficient and accurate than the traditional methods, including the Tikhonov regularization method, the Landweber method and the conjugate gradient method.

Mathematics subject classification: 45Q05, 65L50.
Key words: Inverse black body radiation problem, Variational EM method.

1. Introduction

The inverse black body radiation (BBR) problem is to determine the area temperature distribution subject to the total power spectral measurements of its radiation. The first formulation of the problem was proposed by Bojarski [1] by using the Laplace transform and an iterative process. Since then, several kinds of methods have been developed, see, e.g., [2–6].

Mathematically, BBR is an inherently ill-posed problem since it belongs to the Fredholm integral equation of the first kind. Sun and Jaggard [2] and Dou and Hodgson [4] used the Tikhonov regularization technique to overcome the ill-posedness, but they did not choose rule to fix the regularization parameter. Li and Xiao [6] applied the Morozov discrepancy technique to determine the parameter. However, the method works well only when the measurement error is known beforehand. On the other hand, Dou and Hodgson [4] introduced the potential function and applied the entropy method to study the problem. Recently, Li [5] presented some numerical results making use of the conjugate gradient method.

In this work, we propose a variational EM method for BBR problem, which is a variant of well-known EM algorithm [7,8]. Its convergence study is given in the appendix. We compare our method with three traditional methods: the Tikhonov regularization method, the Landweber method and the conjugate gradient method. Numerical experiments demonstrate that the proposed method is more efficient and accurate than the traditional methods.

The organization of the paper is as follows. In Section 2, we introduce the BBR problem and its mathematical formulas. After reviewing the traditional methods in Section 3, we propose a variational EM method for the BBR problem in Section 4. In Section 5, we discuss some relevant issues in the numerical computation. In Section 6, numerical implementation is provided. Finally, the convergence of the variational EM method is studied in the appendix.

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2. Black Body Radiation Problem

Given the area temperature distribution $a(T)$, the total radiated power spectrum $W(v)$ can be expressed by the Planck's law as

$$W(v) = \frac{2hv^3}{c^2} \int_0^\infty \frac{a(T)}{e^{hv/kT} - 1} dT, \tag{2.1}$$

where h is Planck's constant, k is Boltzman's constant, and c is the velocity of light [1]. Set

$$k(v,T) = \frac{2hv^3}{c^2(e^{hv/kT} - 1)}.$$

Consequently,

$$W(v) = \int_0^\infty k(v,T)a(T)dT. \tag{2.2}$$

In practice, the range of T usually goes from 100K to 1000K, and v goes from 0 Hz to 2×10^{14} Hz. In the following, we set $T \in [T_1, T_2]$ and $v \in [V_1, V_2]$, where

$$T_1 = 100\text{K},\ T_2 = 800\text{K},\ V_1 = 0 \text{ Hz},\ V_2 = 2 \times 10^{14} \text{ Hz}. \tag{2.3}$$

Therefore, the problem can be transformed into a standard Fredholm integral equation of the first kind

$$(Ka)(v) = \int_{T_1}^{T_2} k(v,T)a(T)dT = W(v). \tag{2.4}$$

Here, K is the first kind of integral operator from $L^2[T_1, T_2]$ to $L^2[V_1, V_2]$. Given the limited and sometimes noisy power spectrum $W(v)$, the problem becomes how to determine the area temperature distribution $a(T)$ from Eq. (2.4).

3. Traditional Methods

According to the mathematical theories of the inverse problem [9], we implement three kinds of traditional methods for the BBR problem: the Tikhonov regularization method, the Landweber method and the conjugate gradient method.

The dual operator K^* of K needs to be known for implementing these methods. According to the definition of dual operators, we have

$$(K^*\phi)(T) = \int_{V_1}^{V_2} k(v,T)\phi(v)dv, \tag{3.1}$$

where K^* is an operator from $L^2[V_1, V_2]$ to $L^2[T_1, T_2]$.

The Simpson's quadrature rule is used to obtain the numerical evaluation of K. We discretize the interested domain as

$$\begin{aligned} v_i &= V_1 + i/M * (V_2 - V_1), \quad i = 0, 1, \cdots, M, \\ T_j &= T_1 + j/N * (T_2 - T_1), \quad j = 0, 1, \cdots, N, \end{aligned} \tag{3.2}$$

where N and M are even natural numbers. We replace $(Ka)(v_i)$ by

$$\sum_{j=0}^{N} w_j k(v_i, T_j)a(T_j), \quad w_j = \begin{cases} \frac{1}{3N}, & j = 0 \text{ or } N, \\ \frac{4}{3N}, & j = 1, 3, \cdots, N-1, \\ \frac{2}{3N}, & j = 2, 4, \cdots, N-2, \end{cases} \tag{3.3}$$